A Systems Approach to Turbulence: Two Applications

Workshop on Turbulent Boundary Layers, University of New Hampshire

Ati Sharma

University of Southampton

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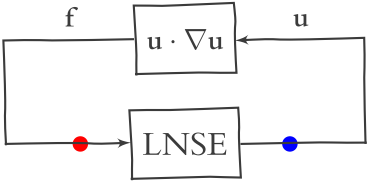
ARC, EPSRC

Introduction

- EXAMPLE: resolvent modelling of 3D lid-driven cavity
- EXAMPLE: passivity-based control of turbulent channel

NSE as a feedback system:

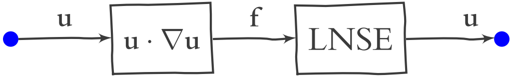
NSE can be represented diagrammatically



Functions $\mathbf{u},\ \mathbf{f} : \Omega \times [0, T] \to \mathbb{R}^3$.

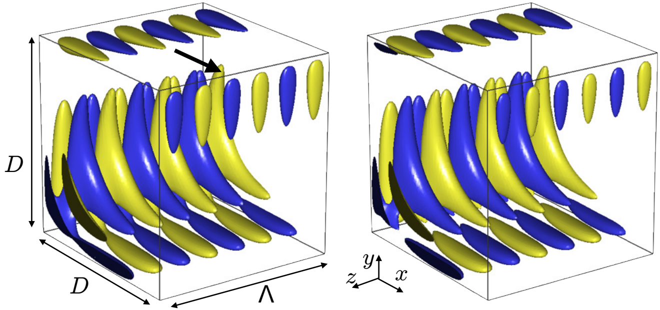
[Aside] Exact solutions

Those that that survive the complete loop unchanged are self-sustaining solutions.



(further aside: this principle is often used to prove stability of closed loop when designing controllers).

Application I: 2D resolvent modes in a lid-driven cavity



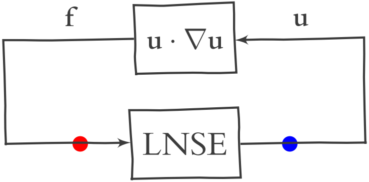
*With F Gomez, M Rudman, H Blackburn (Monash); B McKeon (Caltech)**

* *paper in prep.*

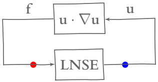
Lid-driven cavity

- $Re = 1200$, $\Lambda = 0.945D$
- Nonlinear, low-dimensional behaviour
- Three dominant wavenumbers: $\beta = 0,\ 3,\ 6$
- Three dominant frequencies: $\omega = 0,\ 0.76,\ 1.52$
- spectral-*hp* 2D $\times$ Fourier (semtex)

Can we apply SVD directly to NSE?



from $\mathbf{f}$ to $\mathbf{u}$ is linear, so yes, in part.



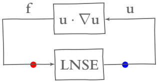
Fourier transform velocity (translation invariant in time, z; neglect transients)

$$\mathbf{u} = \sum_{\beta,\omega} \mathbf{u}_{\beta,\omega}(x, y) \mathrm{e}^{i(\beta z - \omega t)}$$

Assume **time-space mean** $\mathbf{u}_0$ to close.

Same for nonlinear term,

$$-\mathbf{u} \cdot \nabla \mathbf{u} = \sum_{\beta,\omega} \mathbf{f}_{\beta,\omega}(x, y) \ \mathrm{e}^{i(\beta z - \omega t)}$$

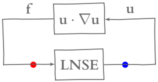


Fluctuations:

$$\mathbf{u}_{\beta,\omega} = (i\omega - \mathcal{L}_\beta)^{-1}\ \mathbf{f}_{\beta,\omega}.$$

Mean:

$$0 = \mathbf{f}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \frac{1}{Re}\nabla^2 \mathbf{u}_0.$$



Take SVD of transfer function,

$$(i\omega - \mathcal{L}_\beta)^{-1} = \sum_m \psi_{\beta,\omega,m} \; \sigma_{\beta,\omega,m} \; \phi^*_{\beta,\omega,m}$$

Gives gain-optimal basis to represent $\mathbf{u}$ and $\mathbf{f}$, scalar coefficient c,

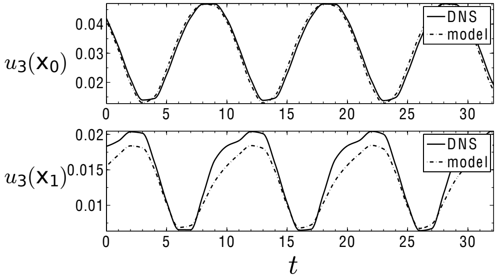
$$\mathbf{u}_{\beta,\omega}(x, y) = \sum_m \psi_{\beta,\omega,m}(x, y) \; c_{\beta,\omega,m}$$

$$\mathbf{f}_{\beta,\omega}(x, y) = \sum_m \phi_{\beta,\omega,m}(x, y) \; c_{\beta,\omega,m} \; / \; \sigma_{\beta,\omega,m}$$

Estimating mode coefficients from probe signal

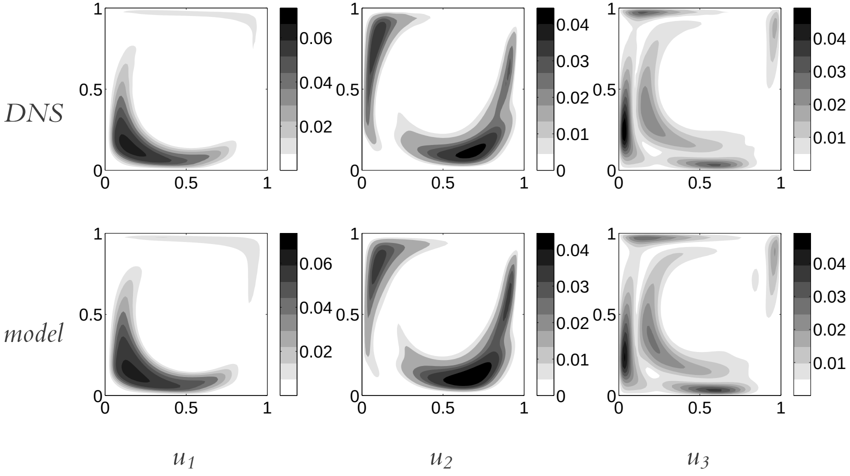
Focus on $\beta = 3$, fit $m = 1$ coefficients at three frequencies.

$$C_\beta = \Psi_\beta^+(\mathbf{x}_p) U_\beta(\mathbf{x}_p)$$



LSQ fit with single probe at $\mathbf{x}_0 = (0.1, 0.1, 0)$*; reconstruction at* $\mathbf{x}_0$ *and* $\mathbf{x}_1 = (0.82, 0.95, 0)$

Reconstructed field (RMS fluctuations)



About 20% model error

isosurfaces at 30% max $w_{\beta=3}$.

Conclusions (I.a)

Observations:

- Quite similar to quasi-linear/RNL but freq-domain; assumes mean
- Solving for mean eqn / coeffs $\simeq$ reintroducing fictitious time

Limitations:

- Needs data to fix amplitudes, phases, dominant ω, β

Conclusions (I.b)

Benefits:

- Meaning of mean flow in linear operators is now clear
- Approximates whole flow from probed points + mean
- Modes are orthogonal (unlike eigenmodes)
- Step towards resolvent models of turbulence in complex geometries

Application II:

Feedback Control in a turbulent channel

Control of a $Re_\tau = 100$ / $Re_{CL} = 2230$ channel

With P Heins, B Jones (Sheffield).*

- Approach is feedback control to stabilise perturbations to fixed point (e.g. laminar)
- Actuation is v-transpiration at the wall
- Sensing is shear stresses at the wall
- Modified J Gibson's *channelflow*

**Heins, Sharma, Jones, UKACC, 2014; & in review.*

Previous work

- Bewley, Kim, Papadakis, KTH and others
- They all used linear LQR / LQG or $\mathcal{H}_\infty$
- Martinelli & al (2011) used comparable approach.

This work follows on from Sharma *et al* 2006, 2009, 2011.

Morrison's group continuing another branch of this work.

Input-output view / Supply rate*

$$\xrightarrow{\mathbf{d}} \boxed{\text{NSE}} \xrightarrow{\mathbf{u}}$$

Consider power supply to the perturbations
$$\mathbf{u} = \mathbf{U}(x,t) - \mathbf{U}_{lam}(x)$$

$$\text{supplied energy} = \int_{\text{time}} \text{force} \times \text{velocity} = -\langle \mathbf{d}, \mathbf{u} \rangle\,.$$

The system G is *passive* if it is only capable of storing and dissipating energy and not producing any of its own.

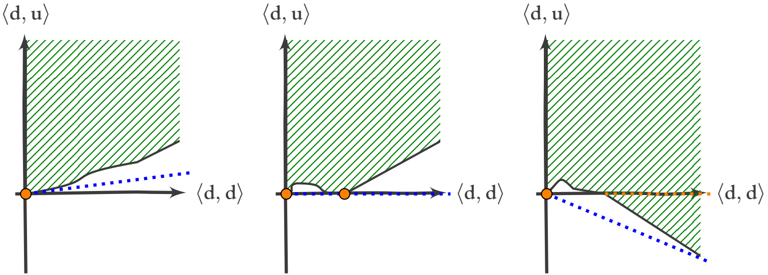
Formally, strictly input passive (SIP) if
$$\langle \mathbf{d}, \mathbf{u} \rangle \geq \varepsilon \langle \mathbf{d}, \mathbf{d} \rangle - \Gamma_0,\ \varepsilon > 0,\ \forall \mathbf{d}.$$

Expresses phase relationship between d and u.
SIP systems are stable.

* *Willems 1972; Zames 1966*

Supply rate and stability

ε bounds net production from above / net dissipation from below.



Laminar solution unique. *Another (energy neutral) solution possible.* *Sustained non-laminar flow possible.*

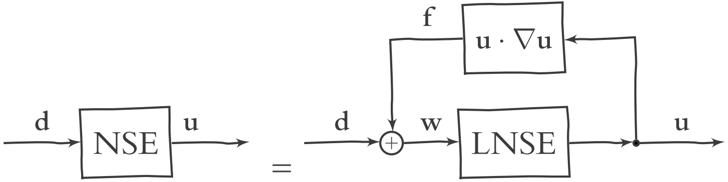
We will use feedback to push up slope of blue line (ε).

Minimise supply rate using actuation

Our game is to find feedback acutation to maximise ε in

$$\langle \mathbf{d}, \mathbf{u} \rangle > \varepsilon \langle \mathbf{d}, \mathbf{d} \rangle$$

with stability of laminar guaranteed if $\varepsilon \geq 0$.



Nonlinear control synthesis problem is actually linear

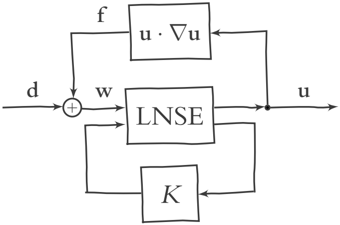
Nonlinearity is conservative,

$$\langle \mathbf{f}, \mathbf{u} \rangle = \langle (\mathbf{u} \cdot \nabla \mathbf{u}), \mathbf{u} \rangle = 0,$$

so $\langle \mathbf{d}, \mathbf{u} \rangle > \varepsilon$ is maximised when $\langle \mathbf{w}, \mathbf{u} \rangle$ is.

Proof is trivial:

$$\langle \mathbf{d}, \mathbf{u} \rangle = \langle \mathbf{w} - \mathbf{f}, \mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle + 0.$$



The optimisation problem

Propose ε; find controller s.t. extremal $\mathbf{d}$ for $\langle \mathbf{d}, \mathbf{u} \rangle = \varepsilon$ leads to variational / TPBVP.

This *nonlinear* variational problem can be solved by *linear* algebraic Riccati eqn matrix methods

(Sun & al 1994).

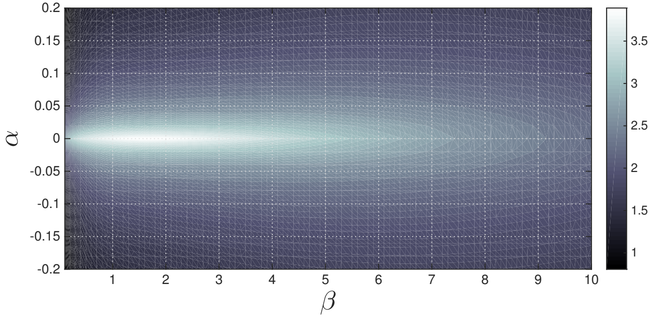
Equivalent to finding optimal feedback to bound

$$\overline{\lambda}(\mathcal{L} + \mathcal{L}^{\dagger})$$

(*c.f. RNL*).

Open-loop energy production, $Re_\tau = 100$

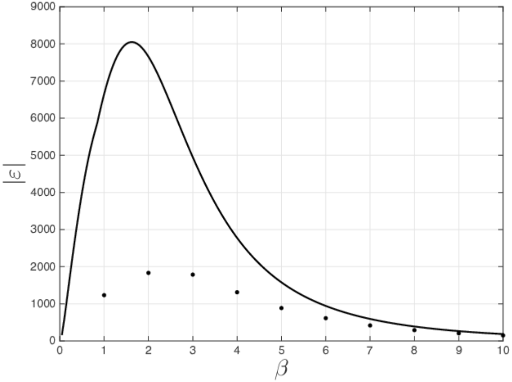
Natural production is concentrated on $\alpha = 0,\ \beta \leq 10$ (*c.f. RNL*)



$log_{10}(\varepsilon)$ in wavenumber space, open-loop

Closed-loop energy production, $Re_\tau = 100$

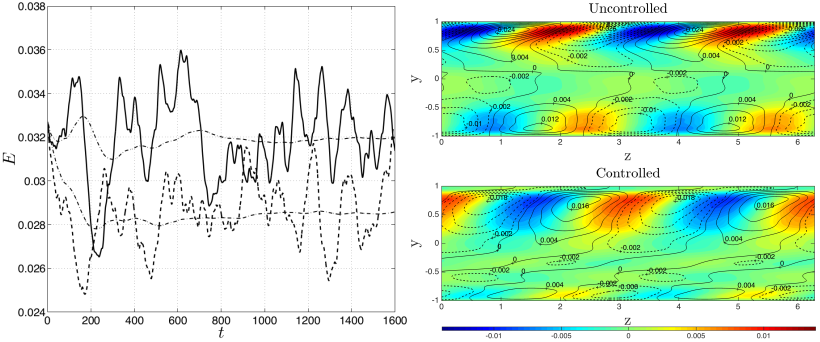
Actuating only on $\alpha = 0,\ \beta \leq 10$.



ε in wavenumber space, closed-loop

Results for $Re_\tau = 100$

Actuating only on $\alpha = 0,\ \beta \leq 10$.



constant U_{bulk}, $4\pi \times 2 \times 2\pi$, $182 \times 151 \times 158$

Conclusions (II)

- Linear mechanisms important:
  - {non-normality; **phase**; lift-up}
  - {pseudo-resonance; gain; critical-layer}
- Reducing gain by feedback does better but is less ‘elegant’.
- Brute-force nonlinear optimisation is expensive and can fail when flow is chaotic (APS G15.00005).

Singular value decomposition

$$M = U\Sigma V^*$$

$$\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ & & & \end{bmatrix}}_{V^*}$$

$$VV^* = V^*V = I, \quad UU^* = U^*U = I$$

$$\sigma_m \geq \sigma_{m+1}$$

σ_m are the *gains*.

Image compression

```
[U, S, V] = svd( img );
aprox_img = U(:,1:m) * S(1:m,1:m) * V(:,1:m)';
```

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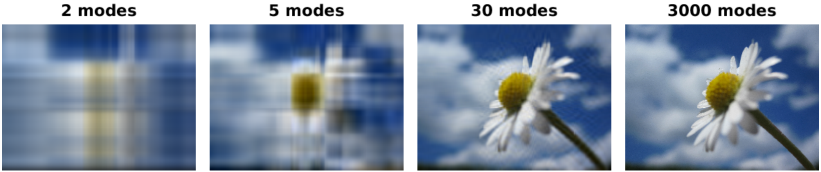

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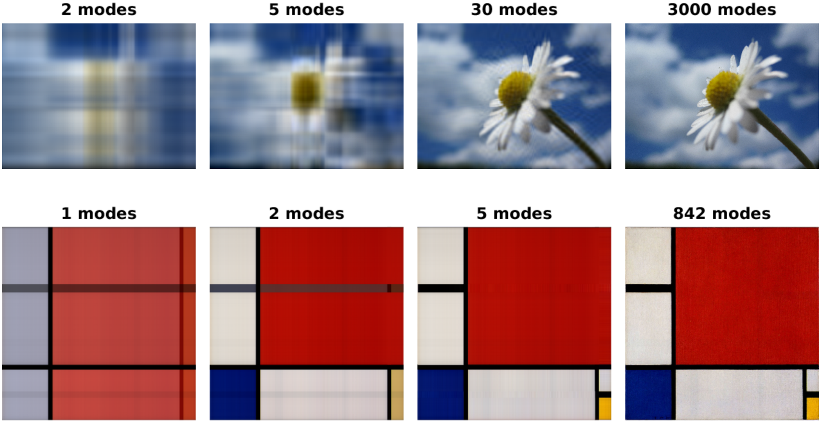

D29.00010

Varadevu, Sharma, Ganapathisubramani

Exact laminar solutions for flows in channels
with sinusoidal walls

"invariant solutions in ω-domain with roughness"

G15.00005

Otero, Sharma, Sandberg

Limitations of Adjoint-Based Optimization
for Separated Flows

"Fully compressible adjoint solver"