Quasi-Linearization of Anisotropic & Inhomogeneous Turbulent Flows

Brad Marston

Department of Physics

Those with Macs running OS X 10.9+ can follow along & run simulations by downloading the app GCM from the Mac App Store

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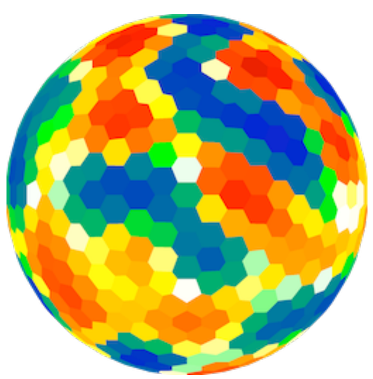
Mac App Store Preview

What's New | What is OS X | OS X Apps | How to Upgrade | Tech Specs

GCM

By Brad Marston

Open the Mac App Store to buy and download apps.



Description

Idealized General Circulation Models (GCMs) of planetary atmospheres, and a stellar tachocline, solved by a variety of methods.

GCM Support

What's New in Version 1.1.5

Introduces a new quasi-linear spectral DNS method, and fixes a bug in the multi-layer spectral DNS method that was introduced in v1.1.3. Also included are some bug fixes to the interface, and improvements to the accuracy of the time integrators.

Free

Category: Education
Updated: Dec 06, 2014
Version: 1.1.5
Size: 1.4 MB
Language: English
Seller: Brad Marston
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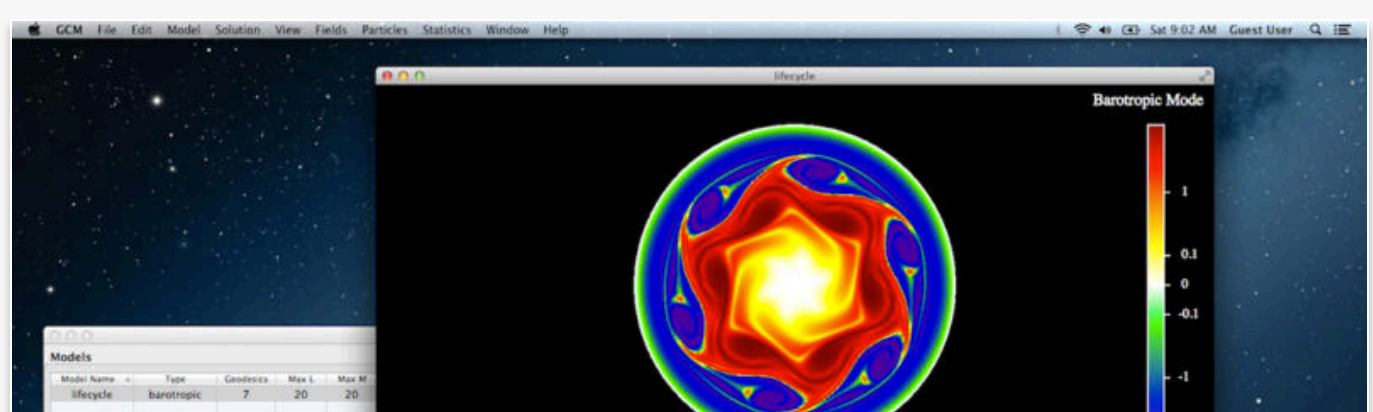
Rated 4+

Compatibility: OS X 10.9.0 or later, 64-bit processor

Customer Ratings

Screenshots



- Introduction to Quasi-Linearization
  - Reynolds decomposition
  - Triadic interactions
  - Numerical Experiments with an Unstable Point Jet
- Direct Statistical Simulation (DSS)
  - Cumulant Expansions
  - Numerical Experiments with a Stochastic Jet
  - Large Deviation Theory
- Generalized Quasi-Linear Approximation (GQL)
  - Wall-Bounded Rotating Couette Flow

Quasi-Linearization

$$\dot{\mathbf{q}} = \mathcal{L}[\mathbf{q}] + \mathcal{Q}[\mathbf{q}, \mathbf{q}]$$

$$\mathbf{q} = \overline{\mathbf{q}} + \mathbf{q}' \qquad \overline{\mathbf{q}'} = 0 \qquad \overline{\overline{\mathbf{q}}} = \overline{\mathbf{q}}$$

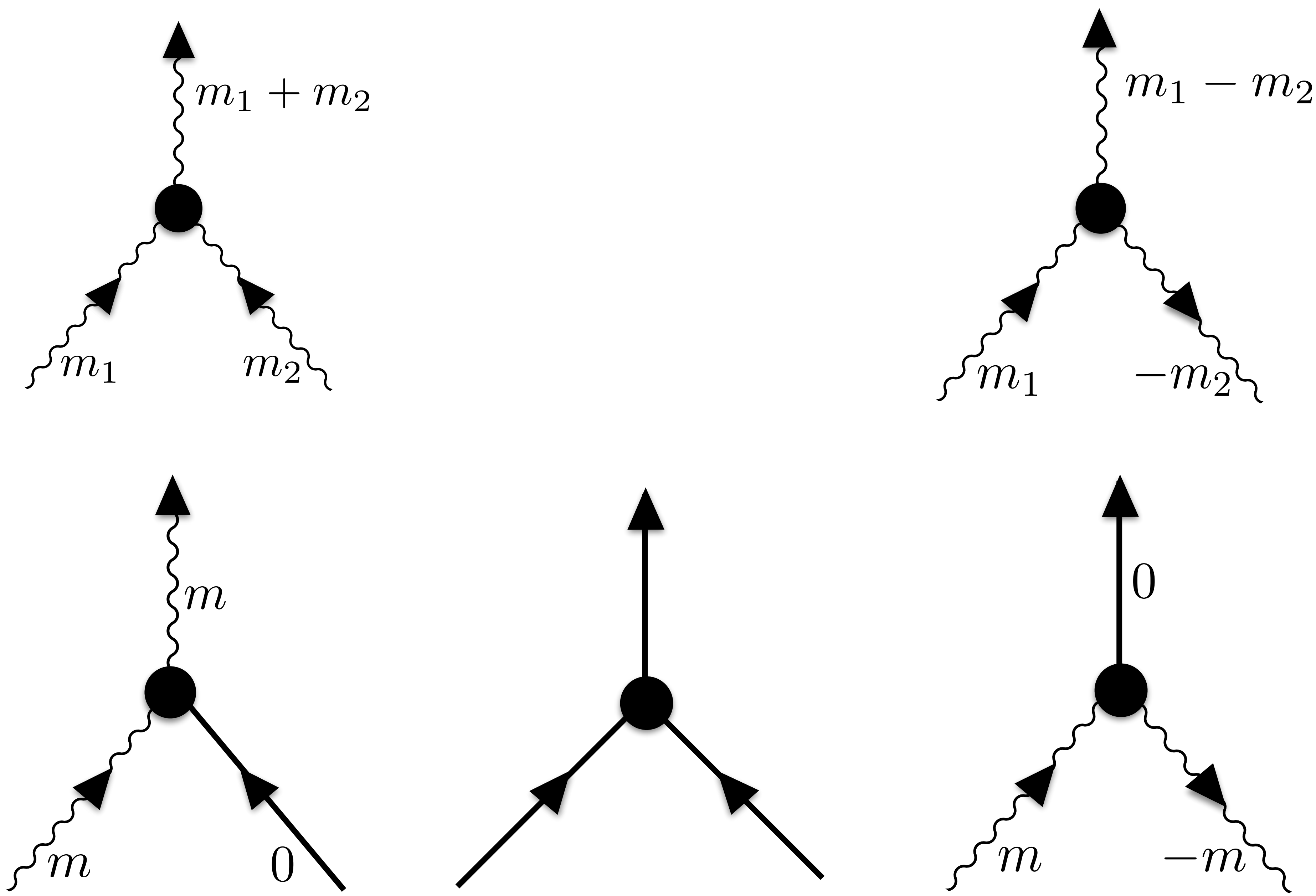
$$\dot{\overline{\mathbf{q}}} = \overline{\mathcal{L}[\mathbf{q}]} + \overline{\mathcal{Q}[\mathbf{q},\ \mathbf{q}]} \quad = \mathcal{L}[\overline{\mathbf{q}}] + \mathcal{Q}[\overline{\mathbf{q}},\ \overline{\mathbf{q}}] + \overline{\mathcal{Q}[\mathbf{q}',\ \mathbf{q}']}$$

$$\dot{\mathbf{q}}' = \mathcal{L}[\mathbf{q}'] + \mathcal{Q}[\overline{\mathbf{q}},\ \mathbf{q}'] + \mathcal{Q}[\mathbf{q}',\ \overline{\mathbf{q}}] + \cancel{\mathcal{Q}[\mathbf{q}',\ \mathbf{q}'] - \overline{\mathcal{Q}[\mathbf{q}',\ \mathbf{q}']}}$$

Drop eddy + eddy —> eddy scattering

Systems with Zonal Symmetry: Zonal Averaging

$$\mathbf{q}(\theta, \phi) = \overline{\mathbf{q}}(\theta) + \mathbf{q}'(\theta, \phi)$$



QL is a conservative approximation obtained by triad decimation

Why Might This Work?

- Quasi-2D: Energy flows upscale
- Heterogeneous shear softens nonlinearities
- Time-scale separation between mean flow and eddies

Herring (1963); O'Gorman and Schneider (2007)

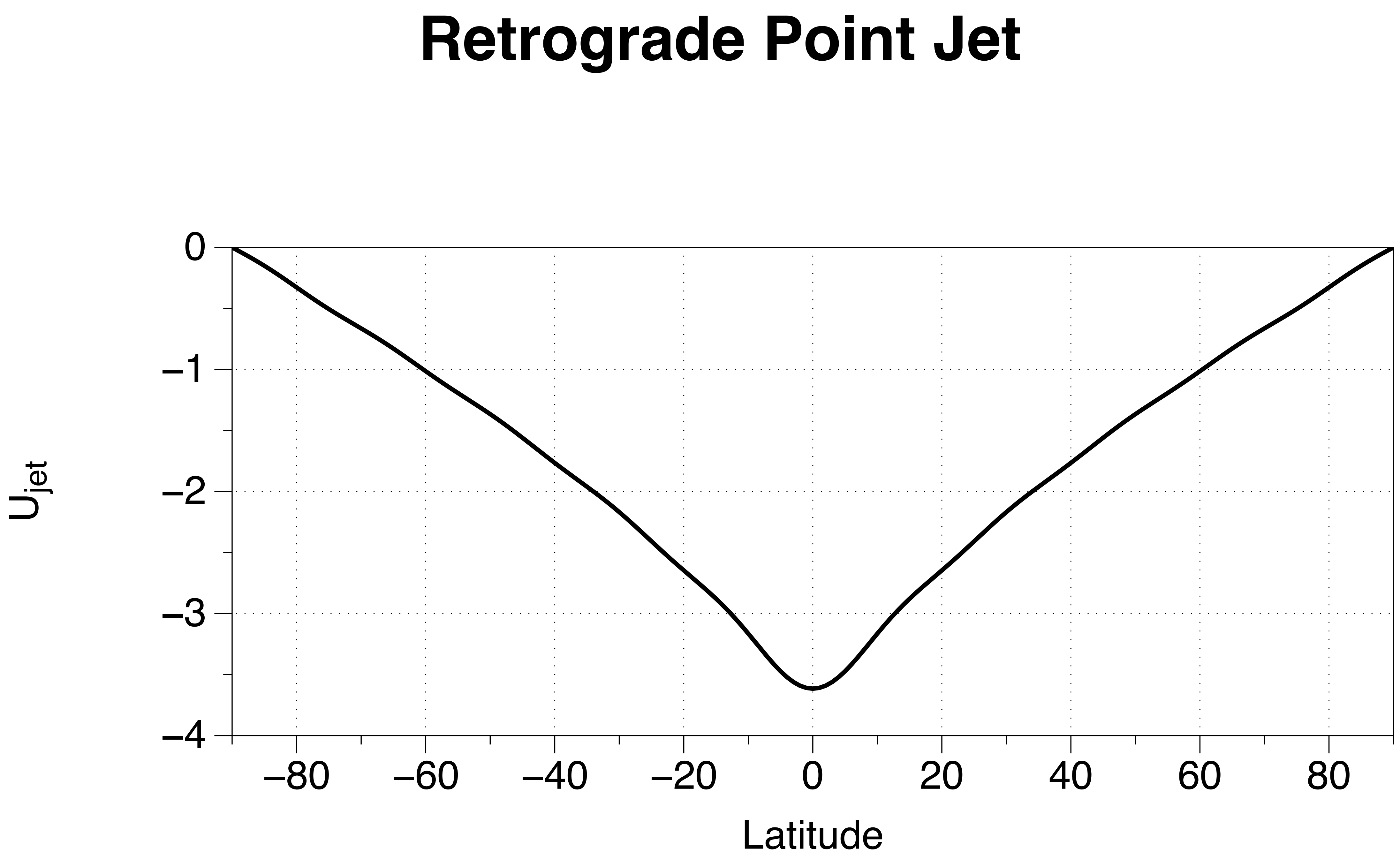
Numerical Experiment: Unstable Barotropic Point Jet

$$\frac{\partial q}{\partial t} + \vec{v} \cdot \vec{\nabla} q = \frac{q_{jet} - q}{\tau}$$

$$q = \omega + f \qquad\qquad f(\theta) = 2\Omega \cos\theta$$

$$\omega = \hat{r} \cdot \vec{\nabla} \times \vec{v} = \nabla^2 \psi$$

$$\vec{v} \cdot \vec{\nabla} \omega = J[\psi,\ \omega] \qquad\qquad J[a,\ b] \equiv \hat{r} \cdot (\vec{\nabla} a \times \vec{\nabla} b)$$



Retrograde Point Jet
0
−1
−2
−3
−4
U_{jet}
−80
−60
−40
−20
0
20
40
60
80
Latitude

"More than any other theoretical procedure, numerical integration is also subject to the criticism that it yields little insight into the problem. The computed numbers are not only processed like data but they look like data, and a study of them may be no more enlightening than a study of real meteorological observations. An alternative procedure which does not suffer this disadvantage consists of deriving a new system of equations whose unknowns are the statistics themselves...."

Edward Lorenz, The Nature and Theory of the General Circulation of the Atmosphere (1967)

“Direct Statistical Simulation” (DSS)

Direct Statistical Simulation (DSS) vs. Direct Numerical Simulation (DNS)

Low-order statistics are smoother in space than the instantaneous flow.

Statistics evolve slowly in time, or not at all, and hence may be described by a fixed point, or at least a slow manifold.

Correlations are *non-local* and highly anisotropic and inhomogeneous. Statistical formulations should respect this.

DSS by Expansion in Equal-Time Cumulants

$$\dot{\mathbf{q}} = \mathcal{L}[\mathbf{q}] + \mathcal{Q}[\mathbf{q}, \mathbf{q}] \qquad \mathbf{q} = \overline{\mathbf{q}} + \mathbf{q}'$$

$$\dot{\overline{\mathbf{q}}} = \overline{\mathcal{L}[\mathbf{q}]} + \overline{\mathcal{Q}[\mathbf{q},\ \mathbf{q}]} \ = \mathcal{L}[\overline{\mathbf{q}}] + \mathcal{Q}[\overline{\mathbf{q}},\ \overline{\mathbf{q}}] + \overline{\mathcal{Q}[\mathbf{q}',\ \mathbf{q}']}$$

$$\overline{\mathbf{q}}(\vec{r}) = c_1(\theta)$$

$$\overline{\mathbf{q}'(\vec{r}_1)\mathbf{q}'(\vec{r}_2)} = c_2(\vec{r}_1, \vec{r}_2) = c_2(\theta_1, \theta_2, \phi_1 - \phi_2)$$

$$\mathbf{q}'(\vec{r}) = \int d\vec{r'}\delta(\vec{r} - \vec{r'})\mathbf{q}'(\vec{r'})$$

$$\overline{\mathcal{Q}[\mathbf{q}'(\vec{r}),\ \mathbf{q}'(\vec{r})]} = \int d\vec{r'}\mathcal{Q}[\delta(\vec{r} - \vec{r'}),\ c_2(\vec{r'}, \vec{r})]$$

"....This procedure can be very effective for problems where the original equations are linear, but, in the case of non-linear equations, the new system will inevitably contain more unknowns than equations, and can therefore not be solved, unless additional postulates are introduced."

Edward Lorenz, The Nature and Theory of the General Circulation of the Atmosphere (1967)

DSS by Expansion in Equal-Time Cumulants

$$\dot{\mathbf{q}}' = \mathcal{L}[\mathbf{q}'] + \mathcal{Q}[\overline{\mathbf{q}},\ \mathbf{q}'] + \mathcal{Q}[\mathbf{q}',\ \overline{\mathbf{q}}]$$

$$\overline{\mathbf{q}'(\vec{r}_1)\mathbf{q}'(\vec{r}_2)} = c_2(\vec{r}_1, \vec{r}_2)$$

$$\dot{c_2}(\vec{r_1}, \vec{r_2}) = 2\big\{\mathcal{L}_1[c_2(\vec{r_1}, \vec{r_2})] + \mathcal{Q}_1[c_1(\vec{r_1}),\ c_2(\vec{r_1}, \vec{r_2})] + \mathcal{Q}_1[c_2(\vec{r_1}, \vec{r_2}),\ c_1(\vec{r_1})]\big\}$$

(a self-consistent Lyapunov equation)

A realizable closure: QL decouples 2nd cumulant from 3rd

$$\overline{\mathbf{q}'(\vec{r}_1)\mathbf{q}'(\vec{r}_2)\mathbf{q}'(\vec{r}_3)} = c_3(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

Investigation of Problems in Thermal Convection

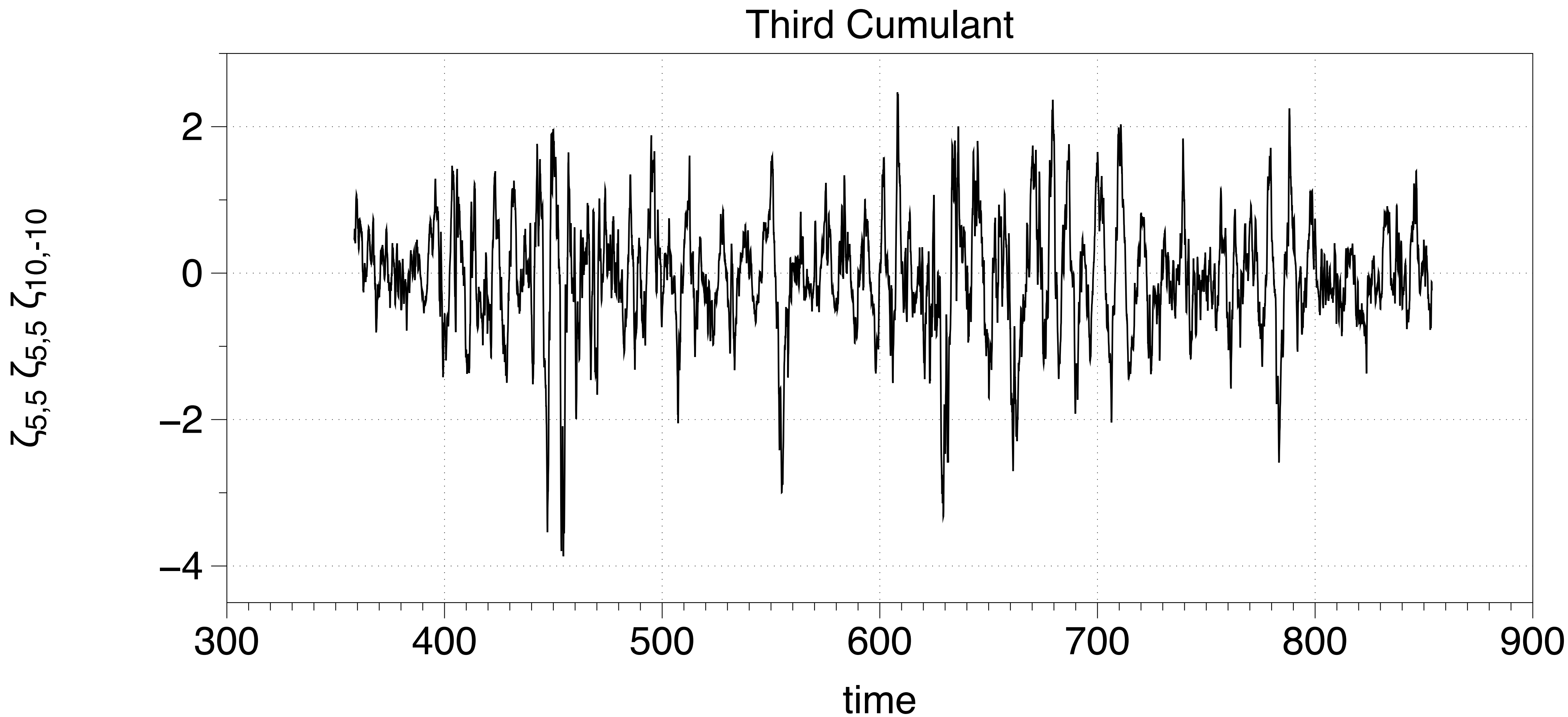
J. R. HERRING

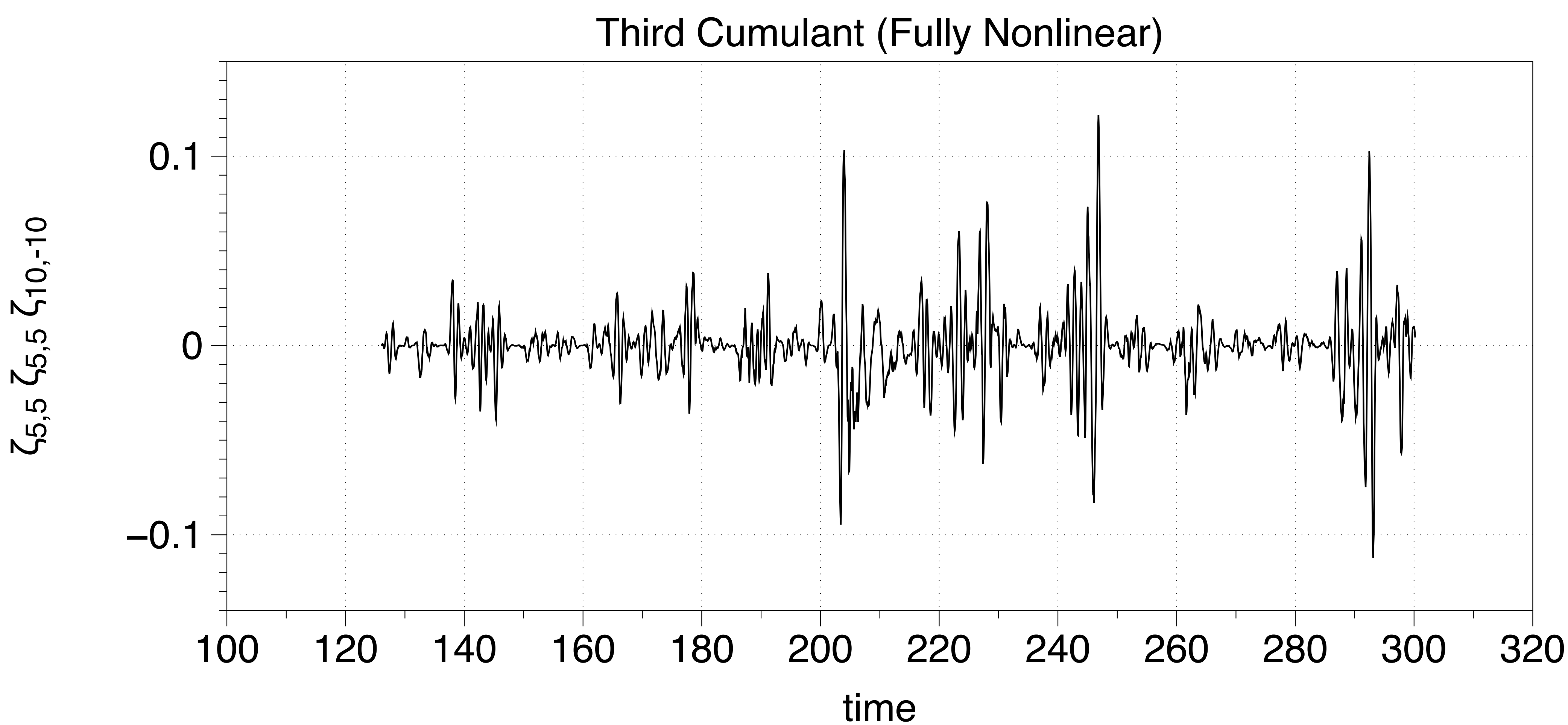
Goddard Institute for Space Studies, New York, N. Y.

(Manuscript received 5 April 1963)

Since the system contains a non-vanishing first-order moment, ψ, the transfer terms contain both correlated third-order moments (cumulants) and products of first order moments with second-order moments. The discarding of the fluctuating self-interaction then corresponds to closing the system of moment equations by discarding the third order cumulants.[2] In the absence of mean fields this procedure would be empty.

[2] Discarding third order cumulants is quite different from discarding third order *moments*. The latter procedure has as a consequence that no steady state nontrivial amplitudes exist. For an investigation of the dynamics of decay for zero third-order moments see Deissler (1962).


Third Cumulant
2
0
−2
−4
300
400
500
600
700
800
900
time
ζ5,5 ζ5,5 ζ10,-10


Third Cumulant (Fully Nonlinear)
0.1
0
−0.1
100
120
140
160
180
200
220
240
260
280
300
320
time
ζ5,5 ζ5,5 ζ10,-10

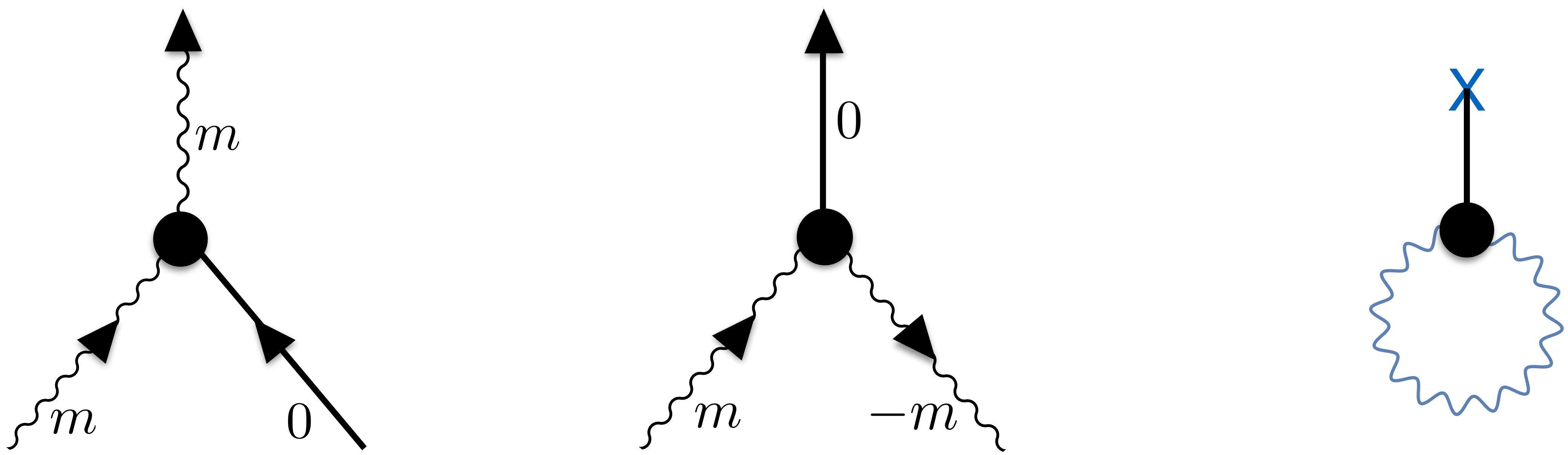
(Short) Time Average for Stochastic PDE

$$\dot{\mathbf{q}} = \mathcal{L}[q] + \mathcal{Q}[q,\ q] + \eta$$

$$\langle \eta(\vec{r_1}, t_1)\ \eta(\vec{r_2}, t_2) \rangle = \Gamma(\vec{r_1}, \vec{r_2})\ \delta(t_1 - t_2)$$

$$\dot{c_2}(\vec{r_1}, \vec{r_2}) = \ldots + \Gamma(\vec{r_1}, \vec{r_2})$$

S3T / CE2

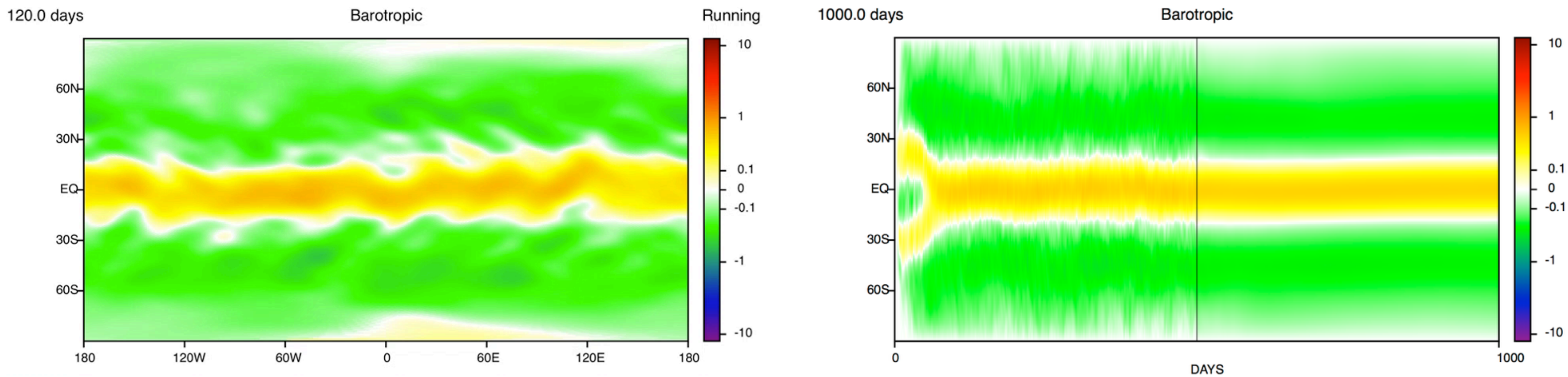


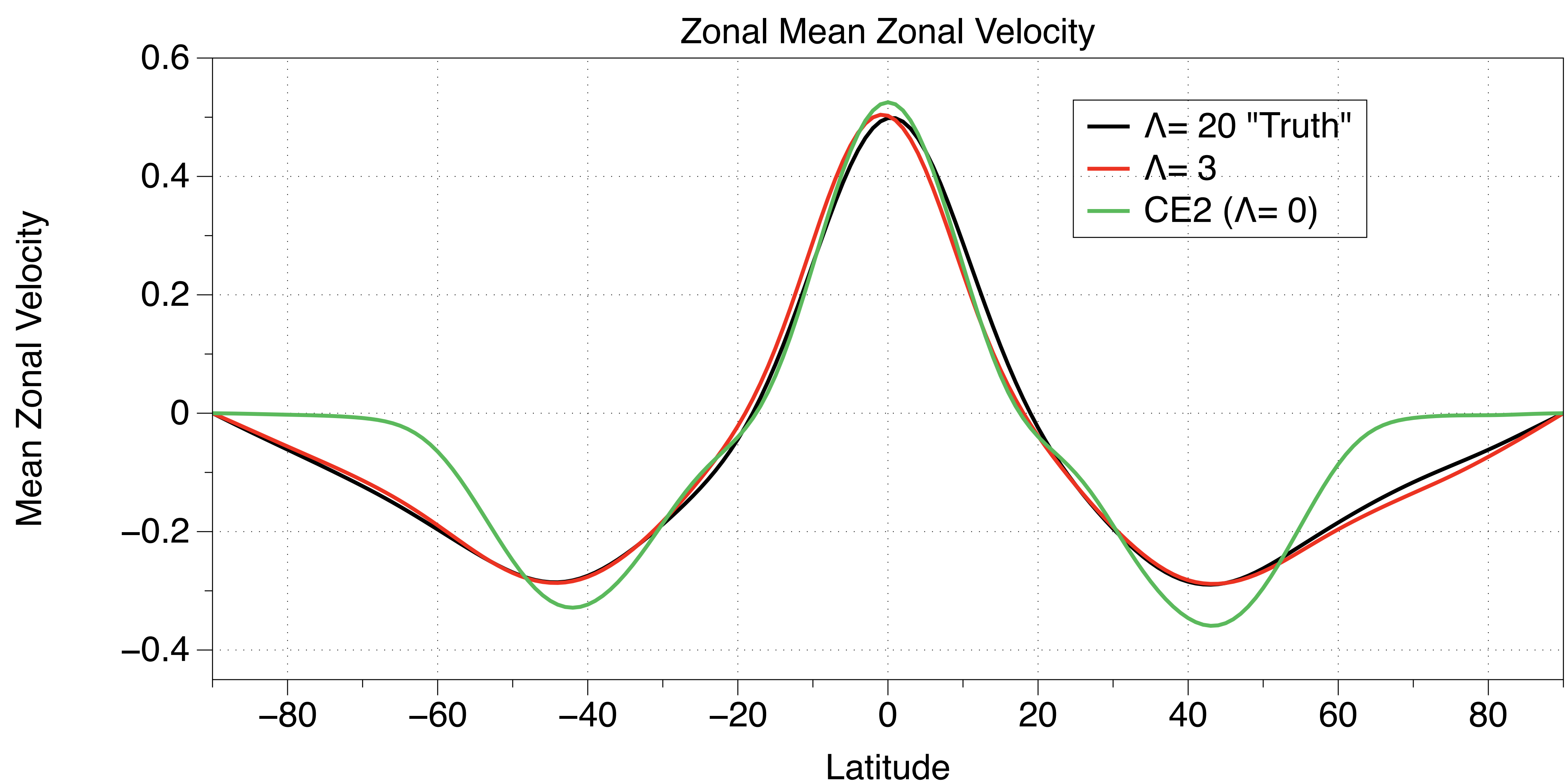
Farrell & Ioannou (2007); JBM, Conover, & Schneider (2008); Bakas & Ioannou (2011); Srinivasan & Young (2012); Parker & Krommes (2013).

JBM, W. Qi, and S. M. Tobias, “Direct Statistical Simulation of a Jet” arXiv:1412.0381 (CE2, CE2.5 and CE3).

Numerical Experiment: Stochastically-Driven Jet

$$\partial_t \zeta + \vec{v} \cdot \vec{\nabla}(\zeta + f) = -\kappa\zeta - \nu_2 \nabla^4 \zeta + \eta$$


120.0 days
Barotropic
Running
60N
30N
EQ
30S
60S
180
120W
60W
0
60E
120E
180
10
1
0.1
0
-0.1
-1
-10
1000.0 days
Barotropic
0
DAYS
1000


Zonal Mean Zonal Velocity
Λ= 20 "Truth"
Λ= 3
CE2 (Λ= 0)
Mean Zonal Velocity
Latitude
0.6
0.4
0.2
0
−0.2
−0.4
−80
−60
−40
−20
0
20
40
60
80

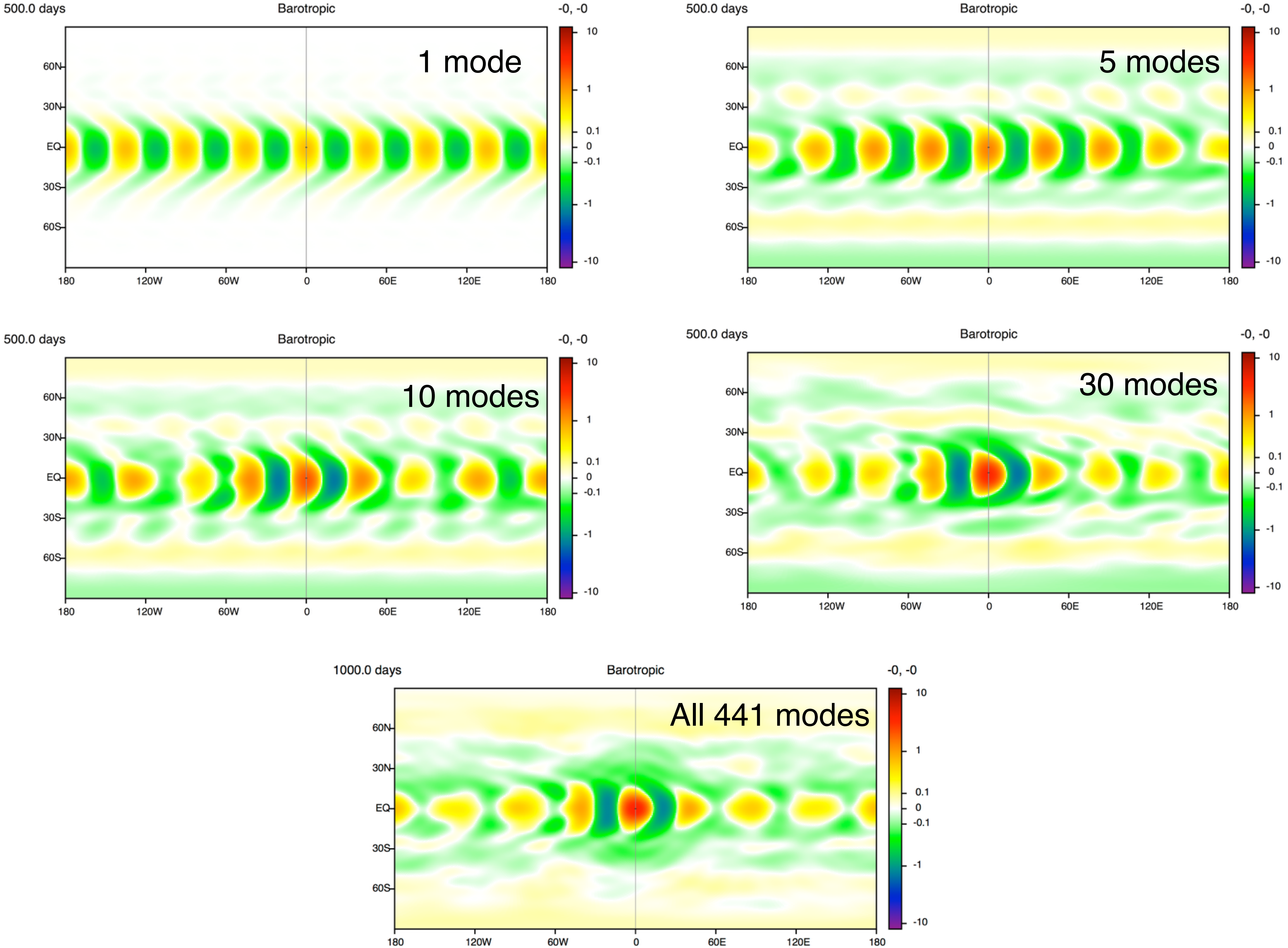
The "Curse of Dimensionality" — How to Address?

Schmidt decomposition

$$\overline{\mathbf{q}'(\vec{r}_1)\mathbf{q}'(\vec{r}_2)} = \sum_i \lambda_i \; \varphi_i(\vec{r}_1) \; \varphi_i(\vec{r}_2); \quad \lambda_i \geq 0$$

$$\approx \sum_{\lambda_i > \lambda_c} \lambda_i \;\; \varphi_i(\vec{r}_1)\varphi_i(\vec{r}_2)$$

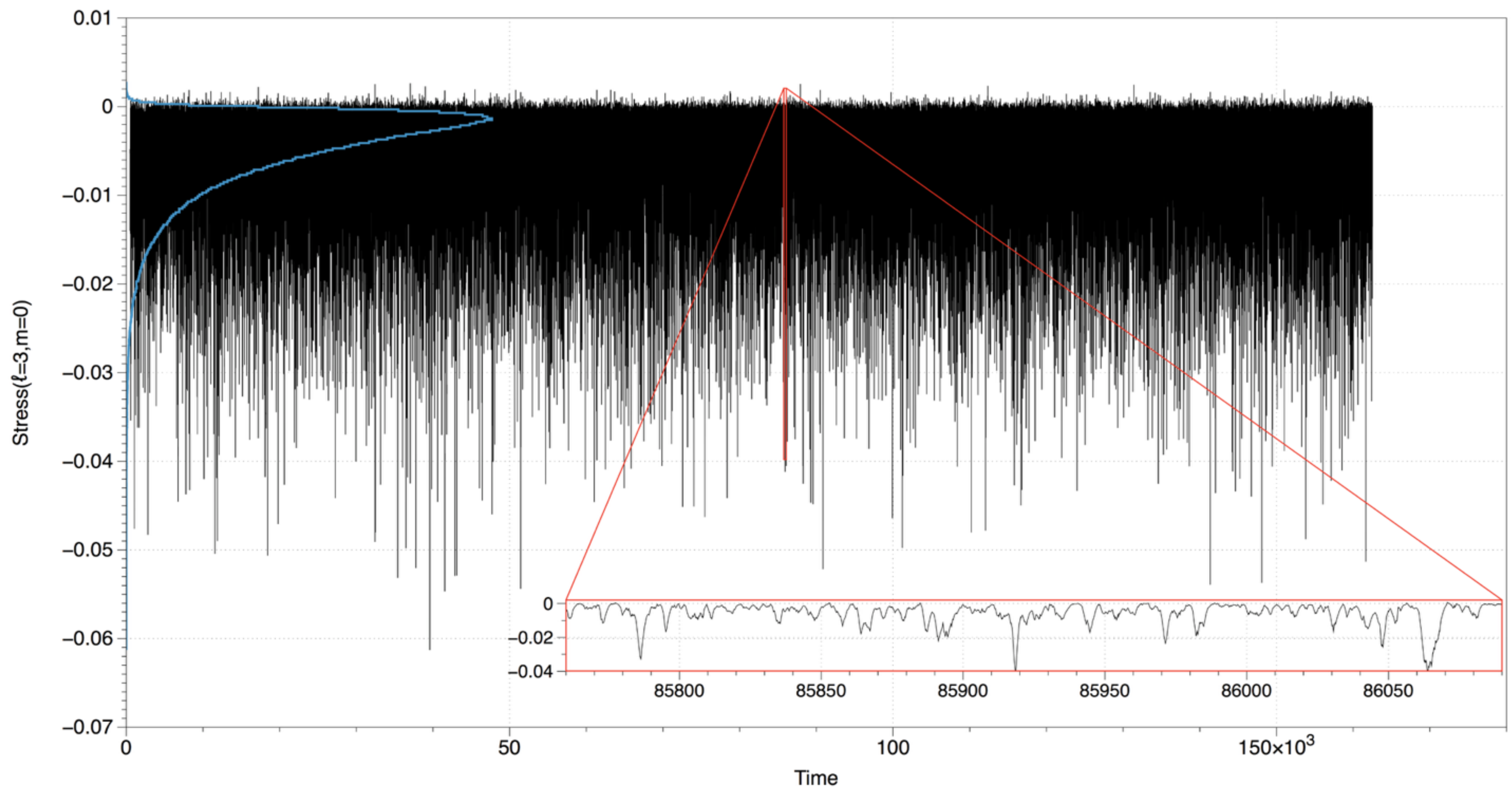
Entanglement:
More than one non-zero eigenvalue

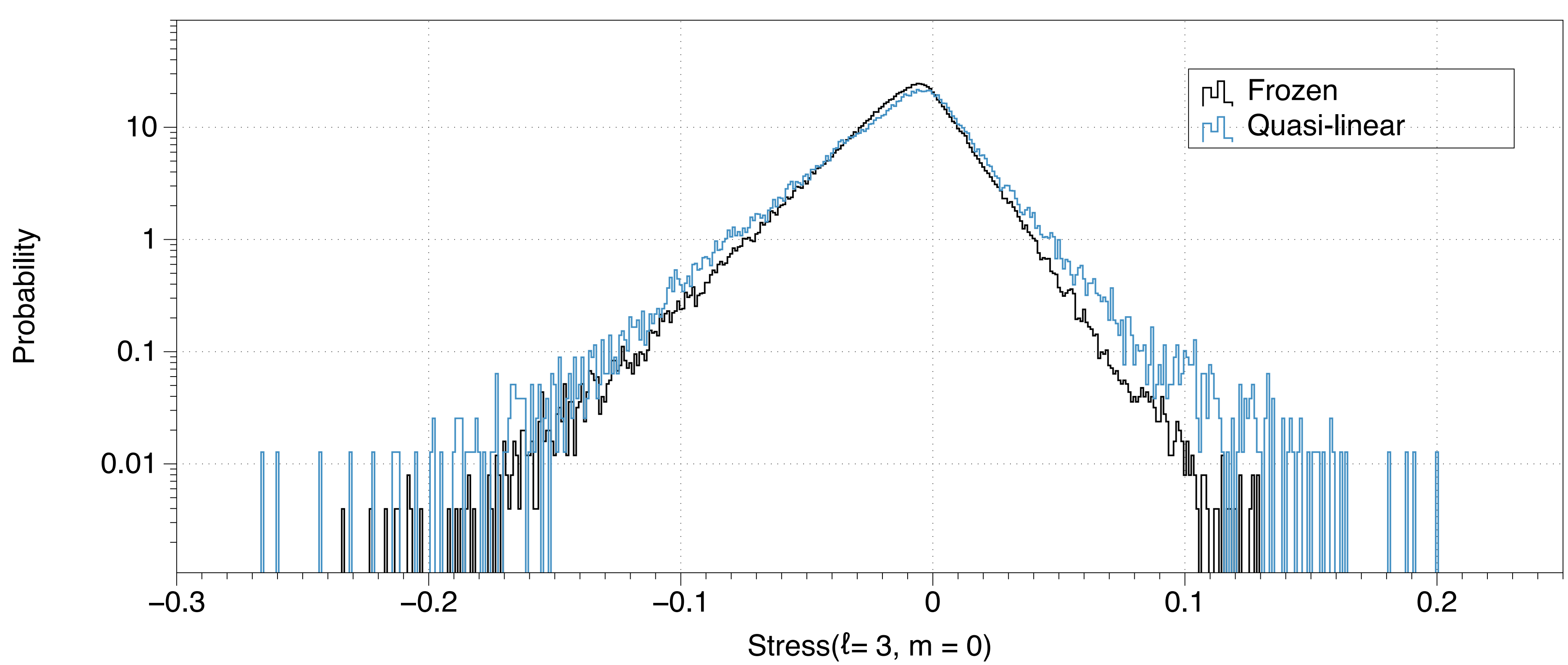


500.0 days
Barotropic
-0, -0
1 mode
500.0 days
Barotropic
-0, -0
5 modes
500.0 days
Barotropic
-0, -0
10 modes
500.0 days
Barotropic
-0, -0
30 modes
1000.0 days
Barotropic
-0, -0
All 441 modes
60N
30N
EQ
30S
60S
180
120W
60W
0
60E
120E
180
10
1
0.1
0
-0.1
-1
-10

Large Deviation Theory

(with Tomás Tangerife and Freddy Bouchet)





Large Deviation Theory

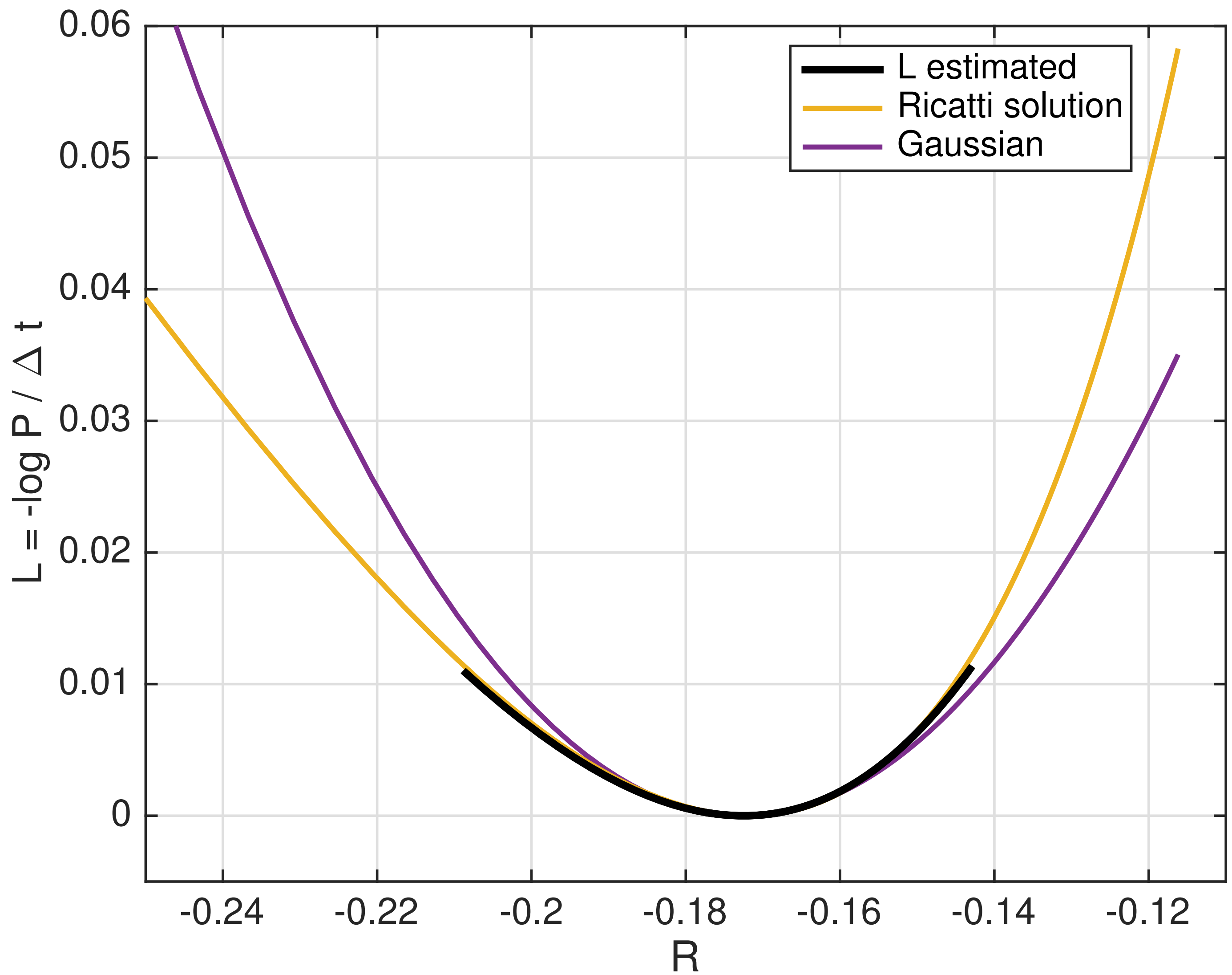
(with Tomás Tangerife and Freddy Bouchet)

$$H_m\left[\theta\right] = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \log \mathbb{E}_U \exp\left[\int \mathrm{d}\phi \, \cos\phi \, \theta\left(\phi\right) \int_0^{\Delta t} R_m\left(\phi, u\right) \, \mathrm{d}u\right]$$

$$\frac{dN}{dt} + NL_x + L_x^T N = 2NCN + \theta\mathcal{M}$$

"Ricatti Equation"

$$\mathcal{L}\left[\frac{\Delta\omega_z}{\Delta t}\right] = \sup_\theta \left\{\theta \cdot \frac{\Delta\omega_z}{\Delta t} - H[\theta]\right\}$$



Applications of DSS

- Macro turbulence
- Boundary Layers driven by convection / shear
- Sub-grid Scale (SGS) Modeling
- Taylor-Couette / Rotating Couette
- Pipe Flow
- Astrophysics & MHD
- Double-diffusive convection?

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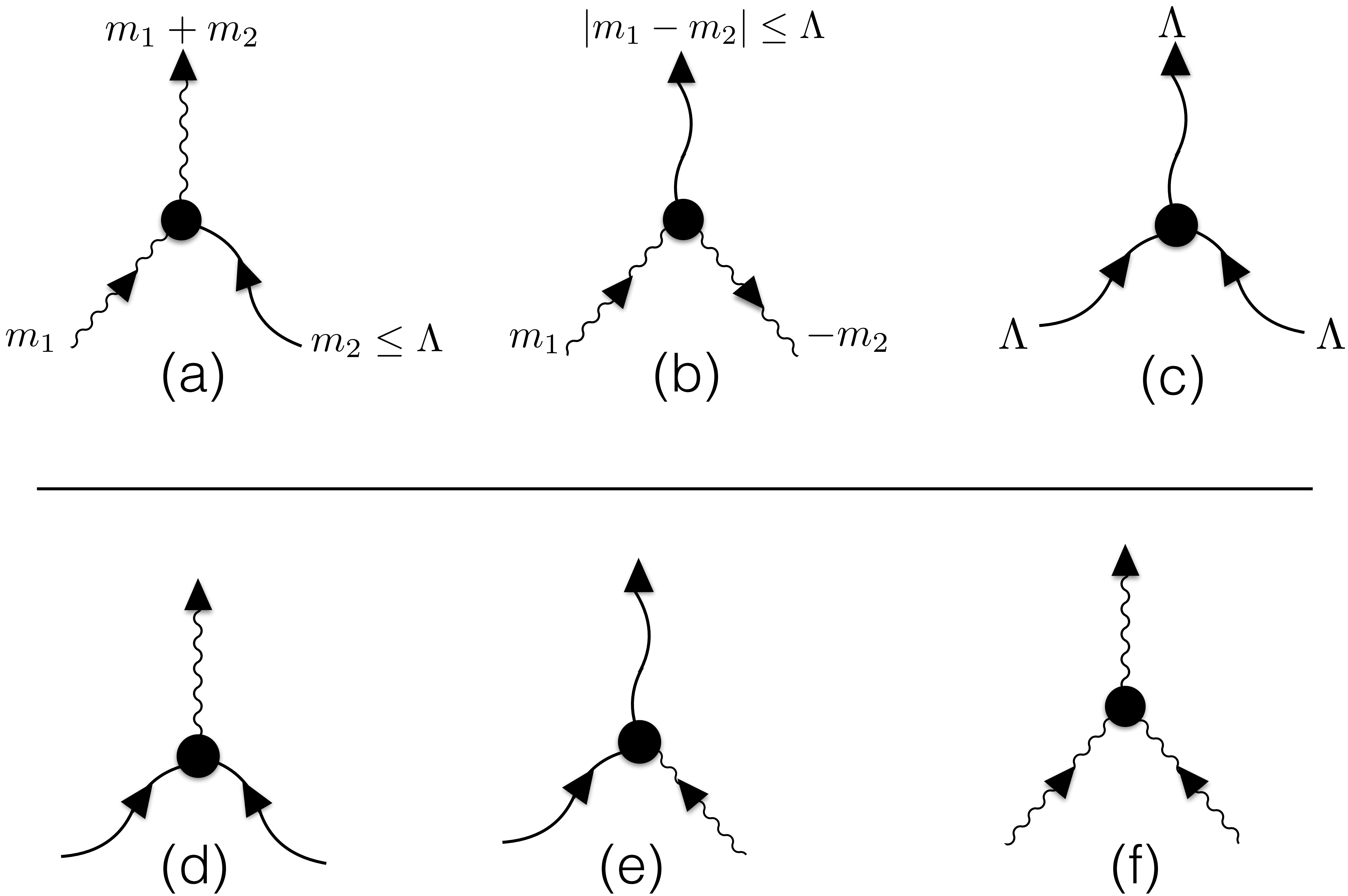
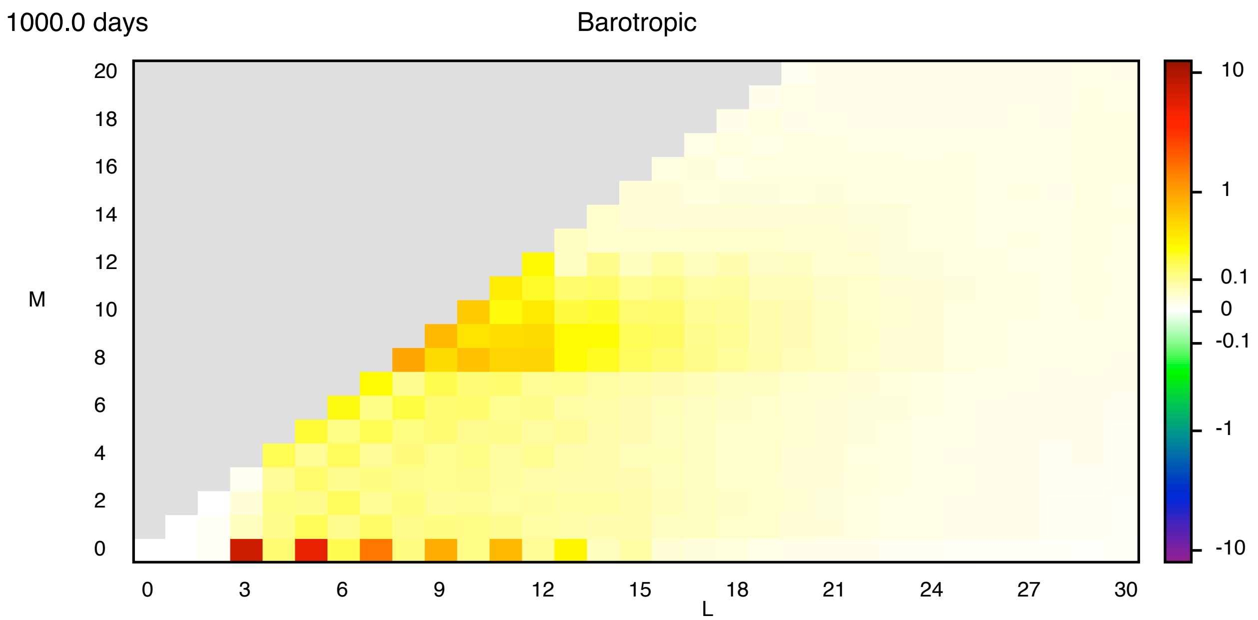
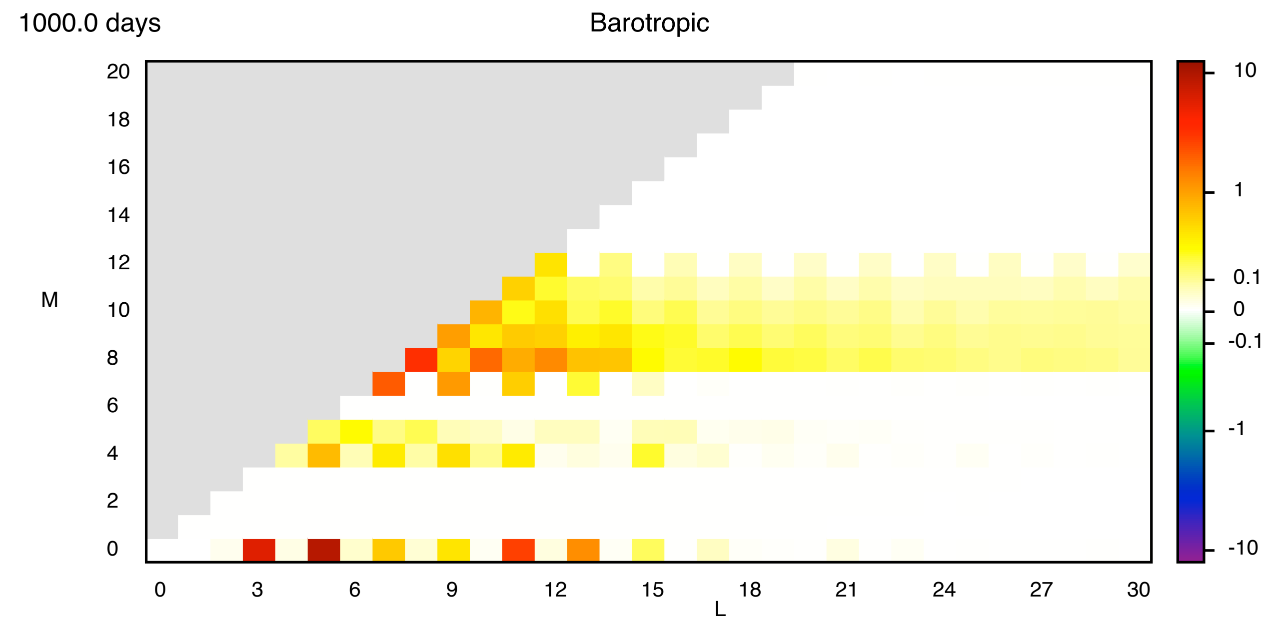
Generalized Quasi—Linear (GQL) Approx.



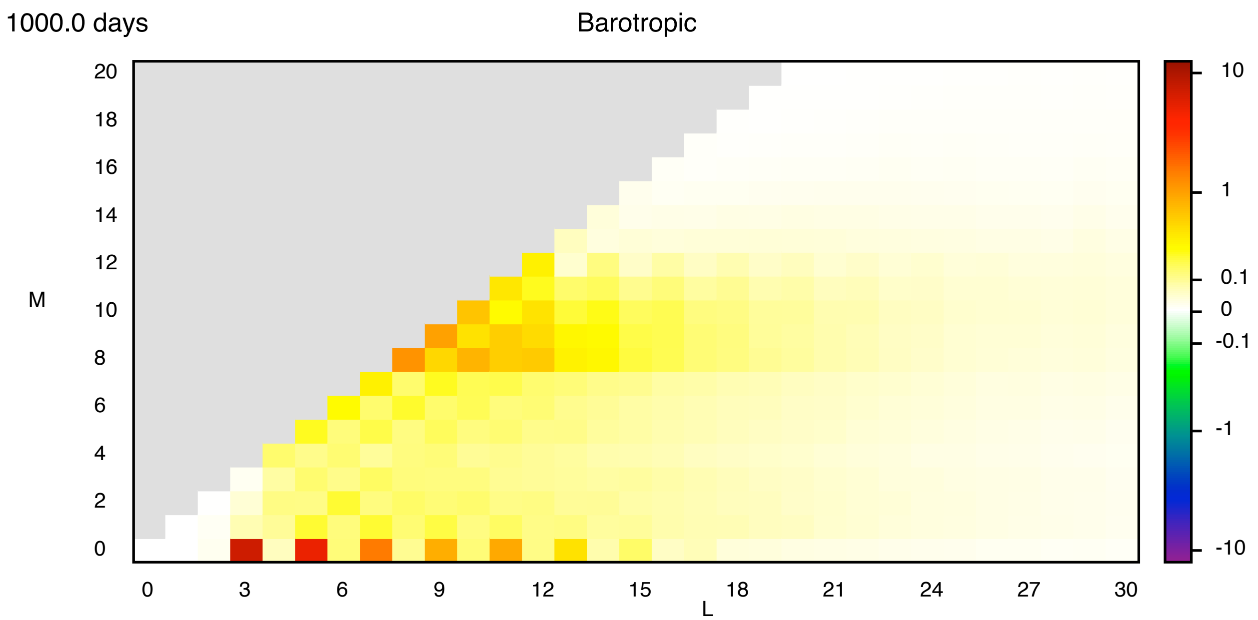
Vorticity Power Spectra



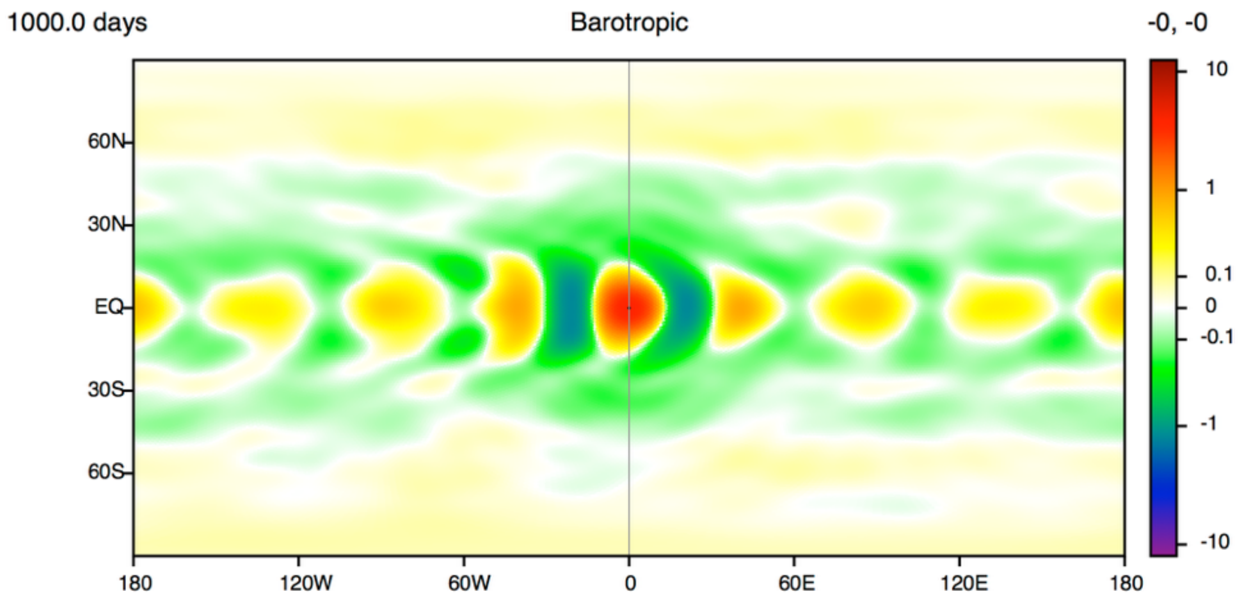
$\Lambda = 20$ “Truth”



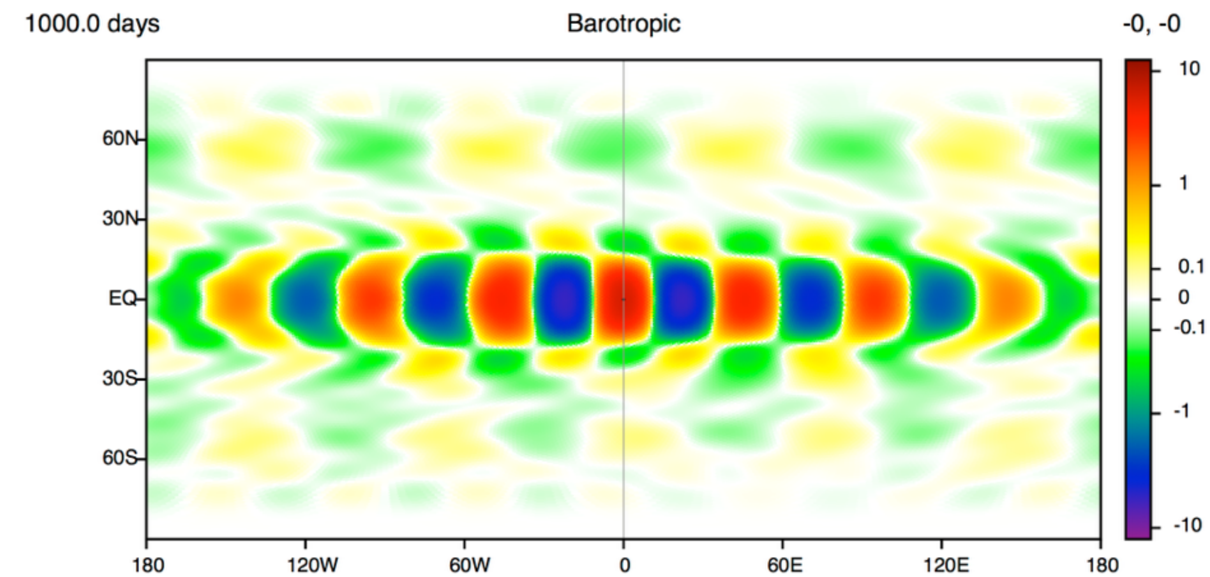
$\Lambda = 0$



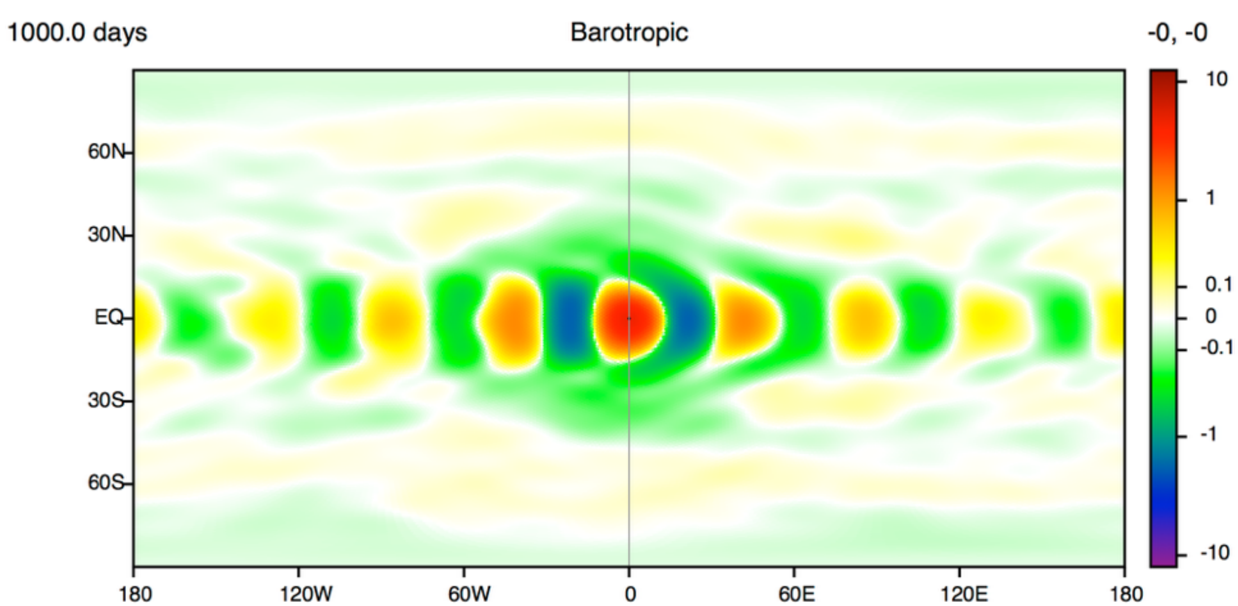
$\Lambda = 3$



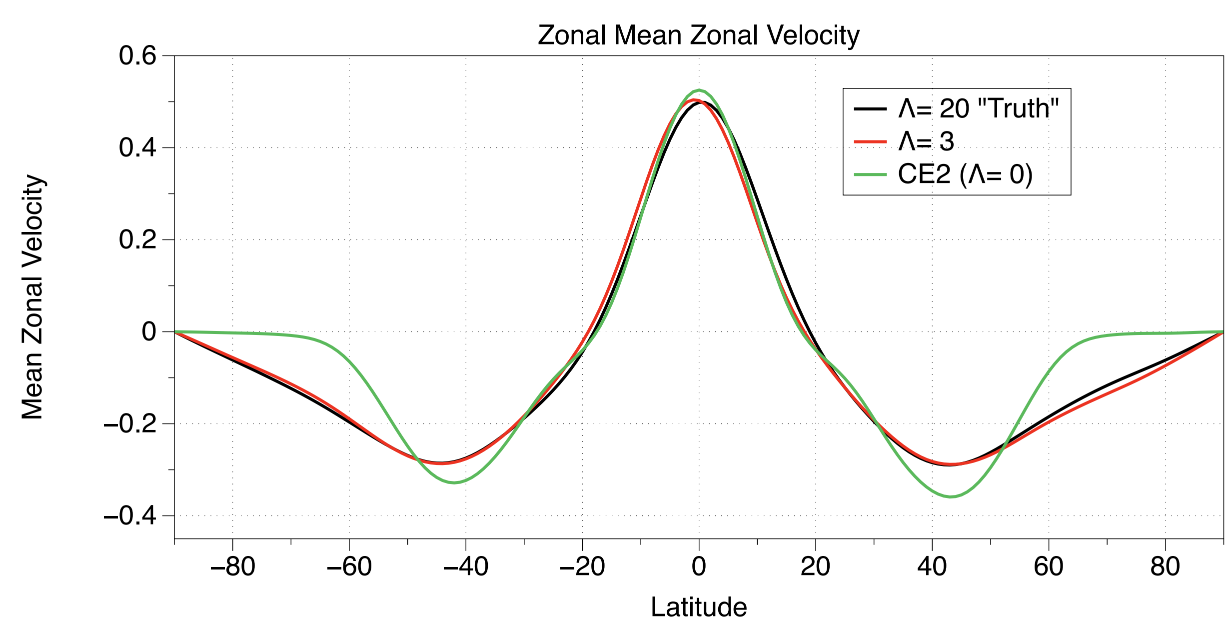
$\Lambda = 20$ “Truth”



$\Lambda = 0$

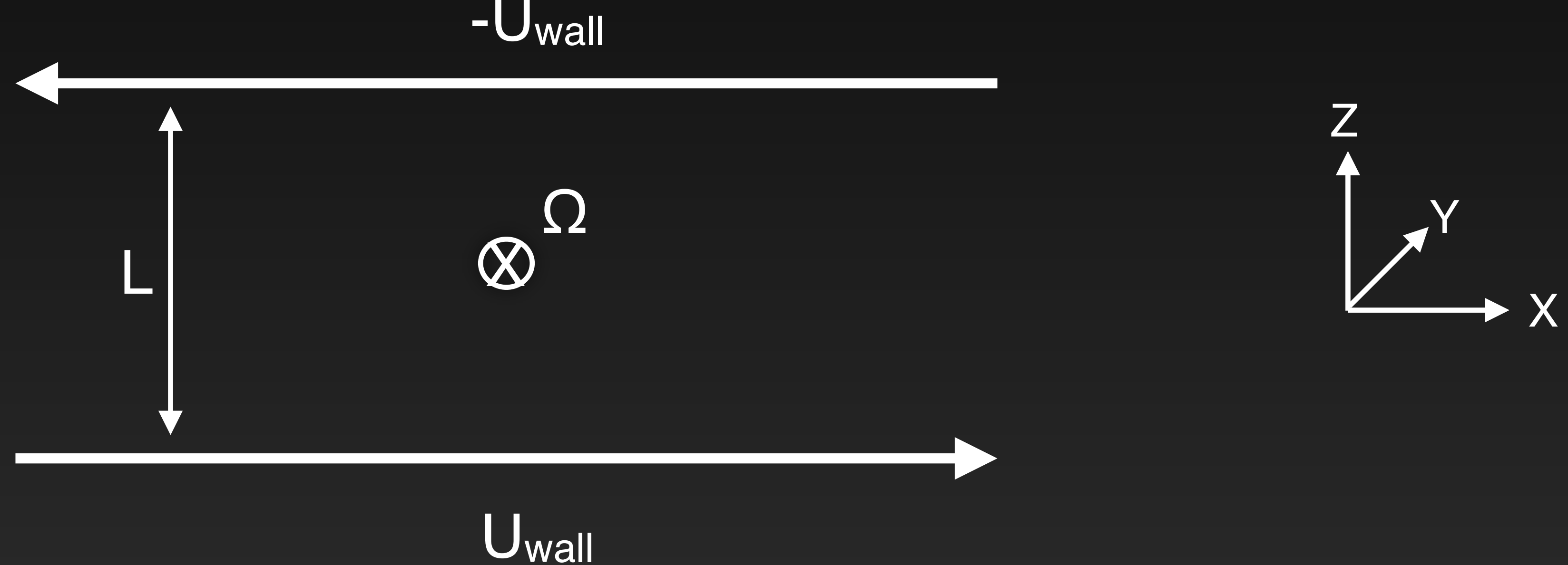


$\Lambda = 3$



Rotating Planar Couette Flow

Bech & Andersson (1996)

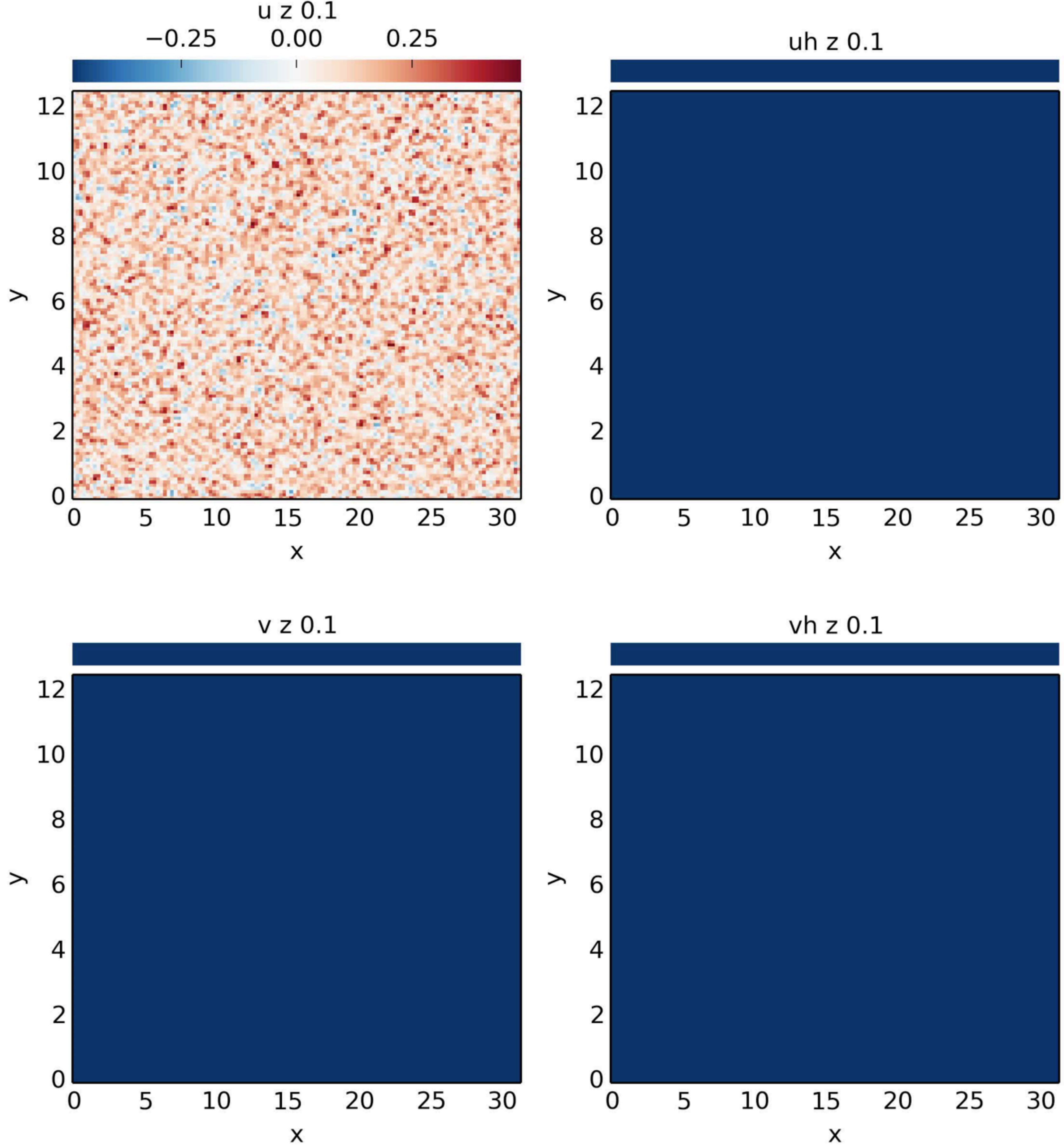


$$R_e = \frac{U_{wall} L}{\nu} = 1,300$$

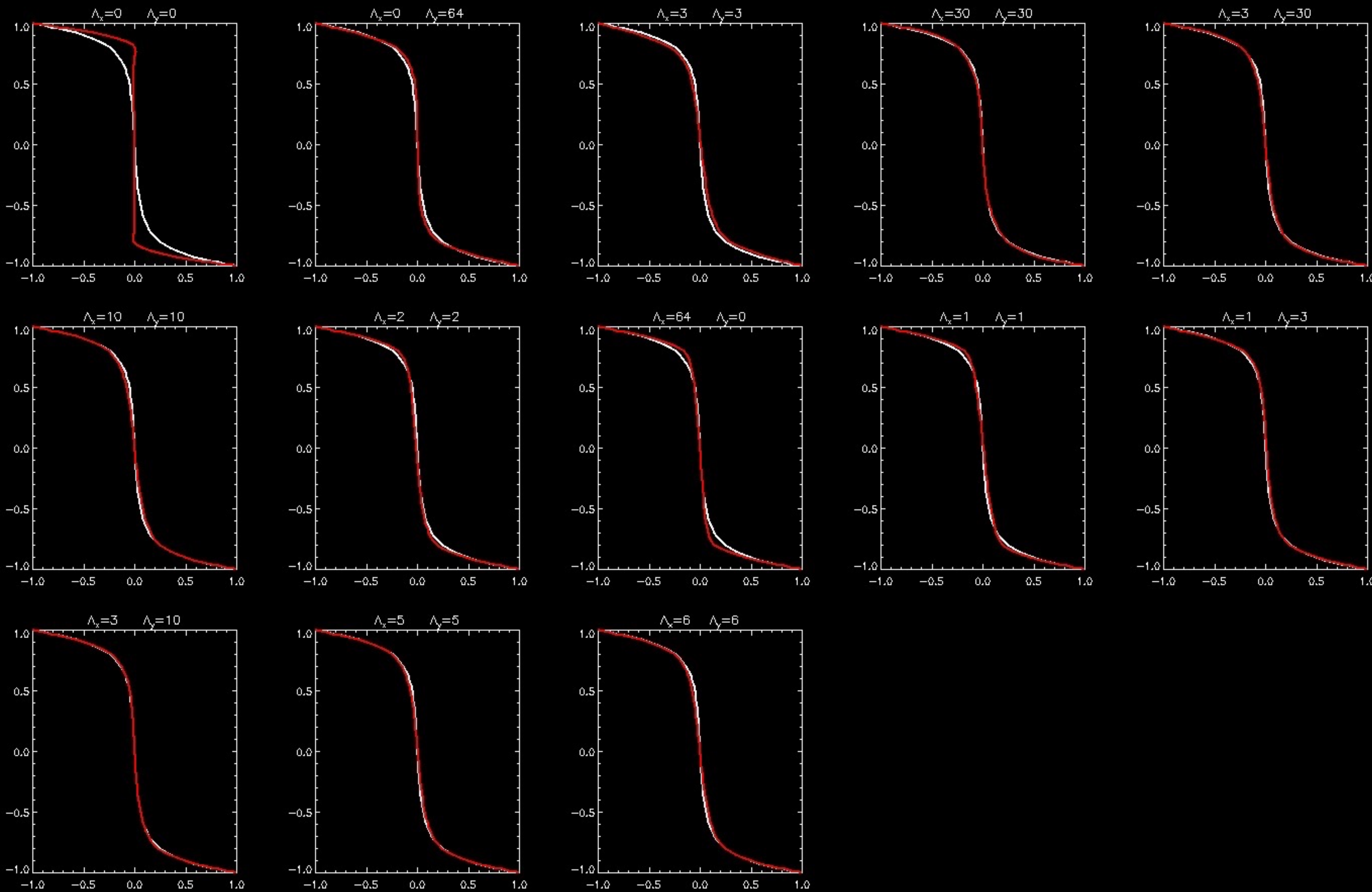
$$R_o = \frac{U_{wall}}{\Omega L} = -100$$

Dedalus Project

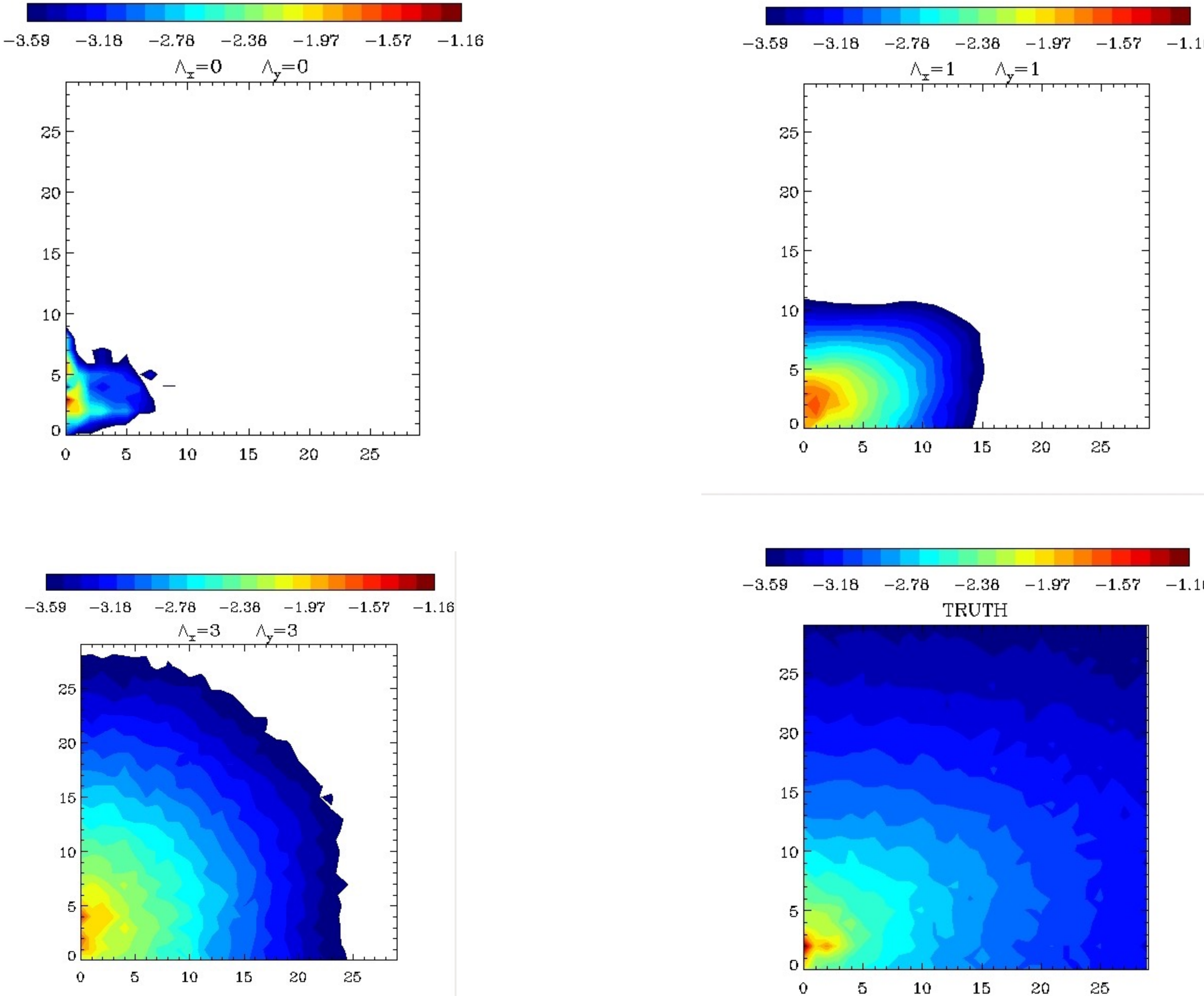
Simulations by Steve Tobias

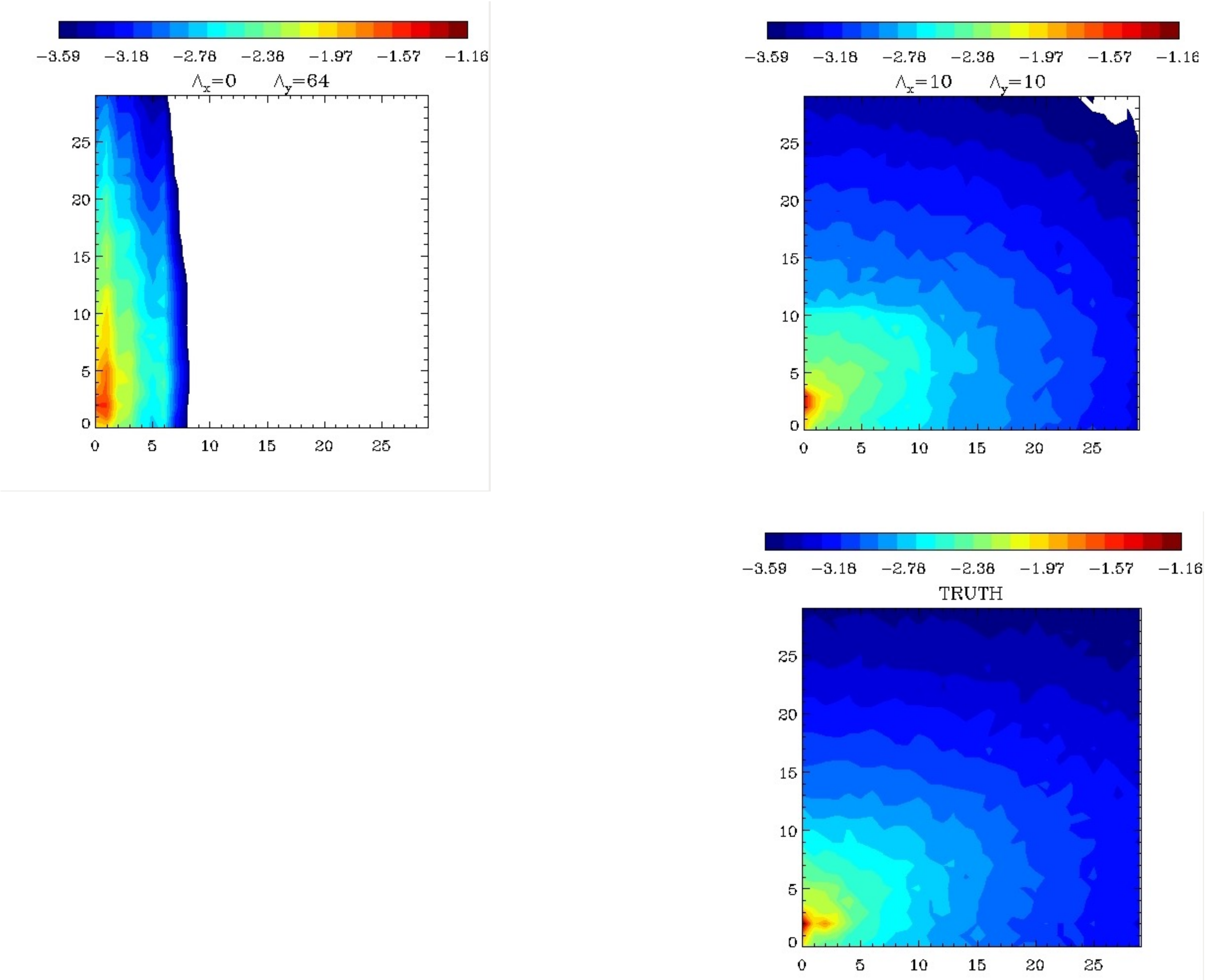


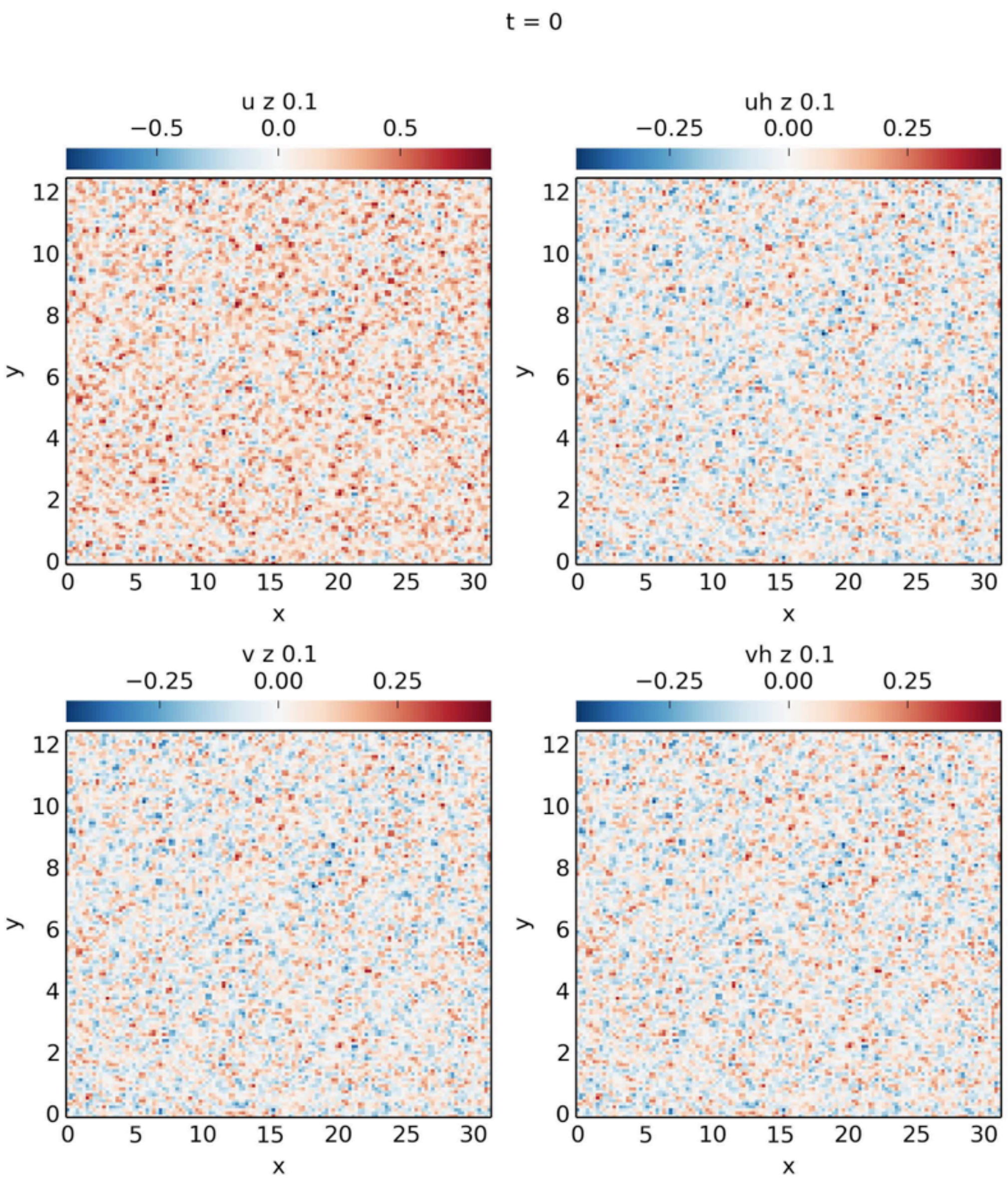
Roll cell instability for rotation antiparallel to mean flow vorticity

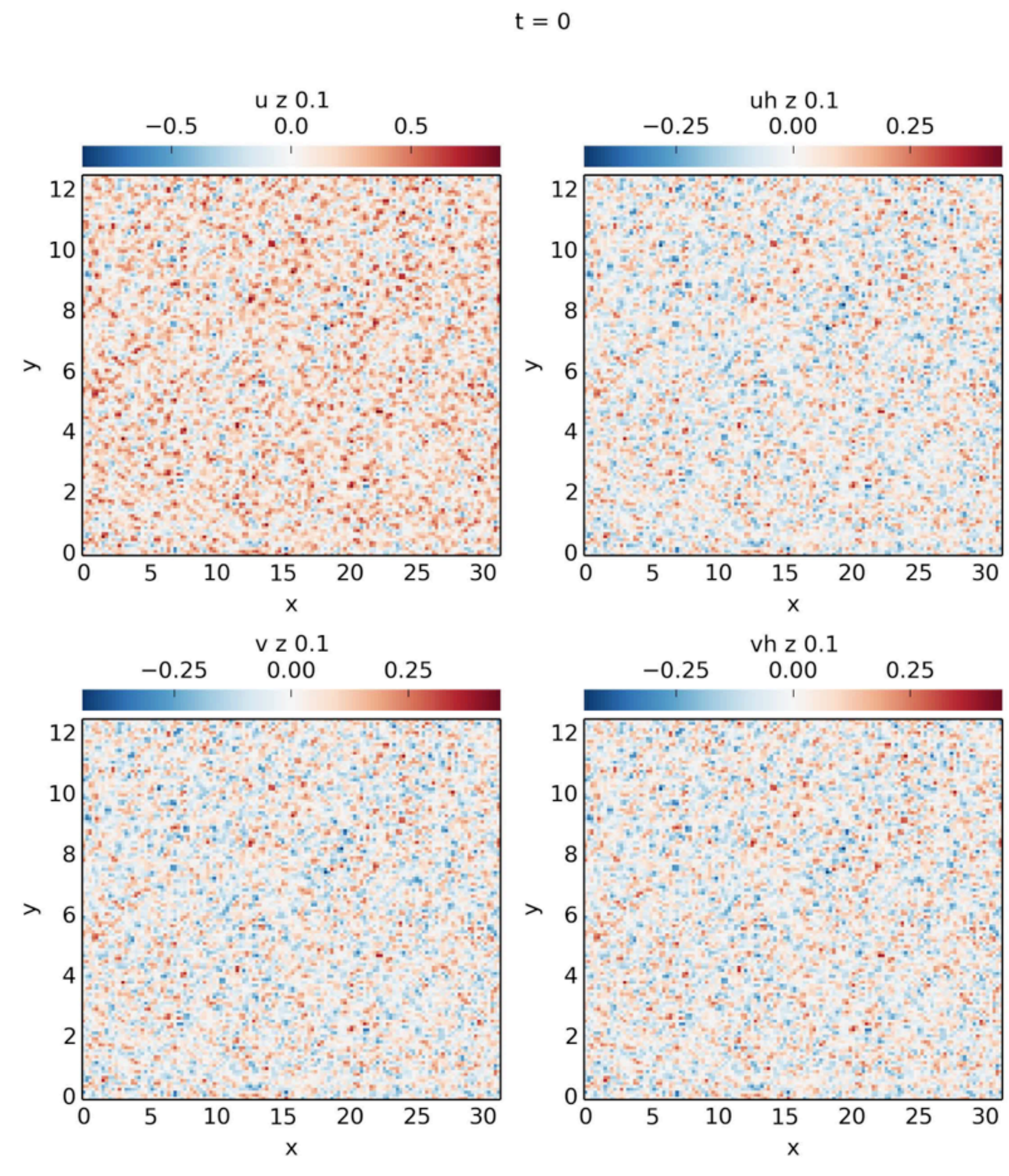


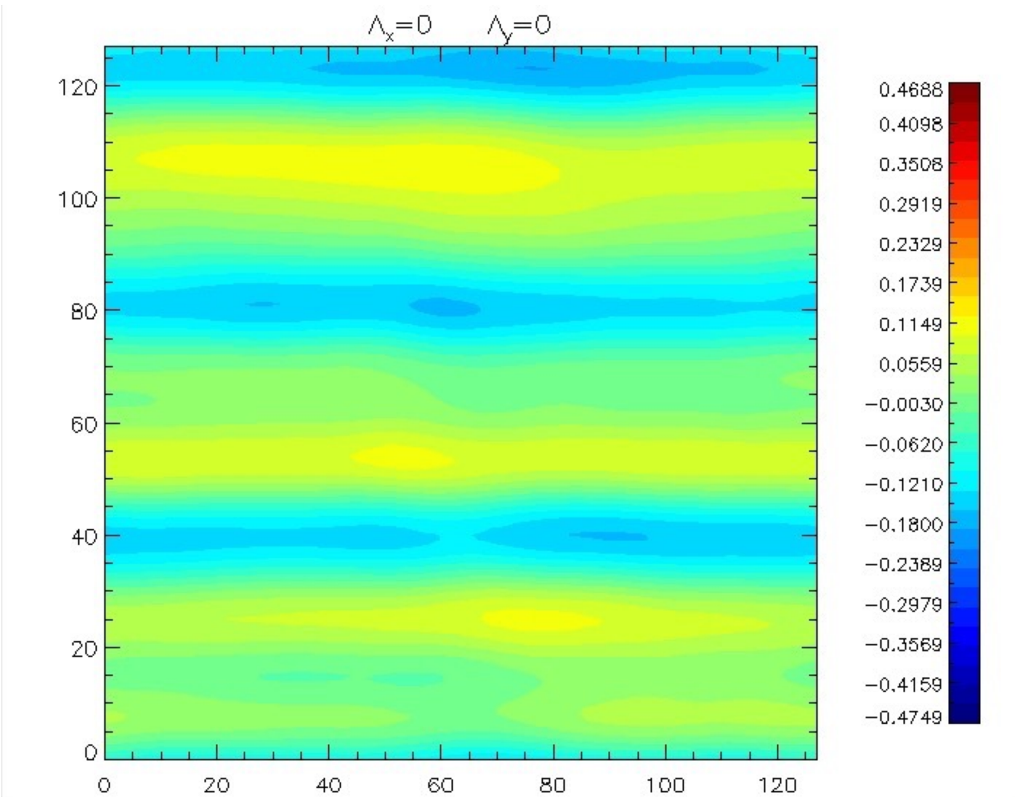
Λx=0 Λy=0
Λx=0 Λy=64
Λx=3 Λy=3
Λx=30 Λy=30
Λx=3 Λy=30
Λx=10 Λy=10
Λx=2 Λy=2
Λx=64 Λy=0
Λx=1 Λy=1
Λx=1 Λy=3
Λx=3 Λy=10
Λx=5 Λy=5
Λx=6 Λy=6

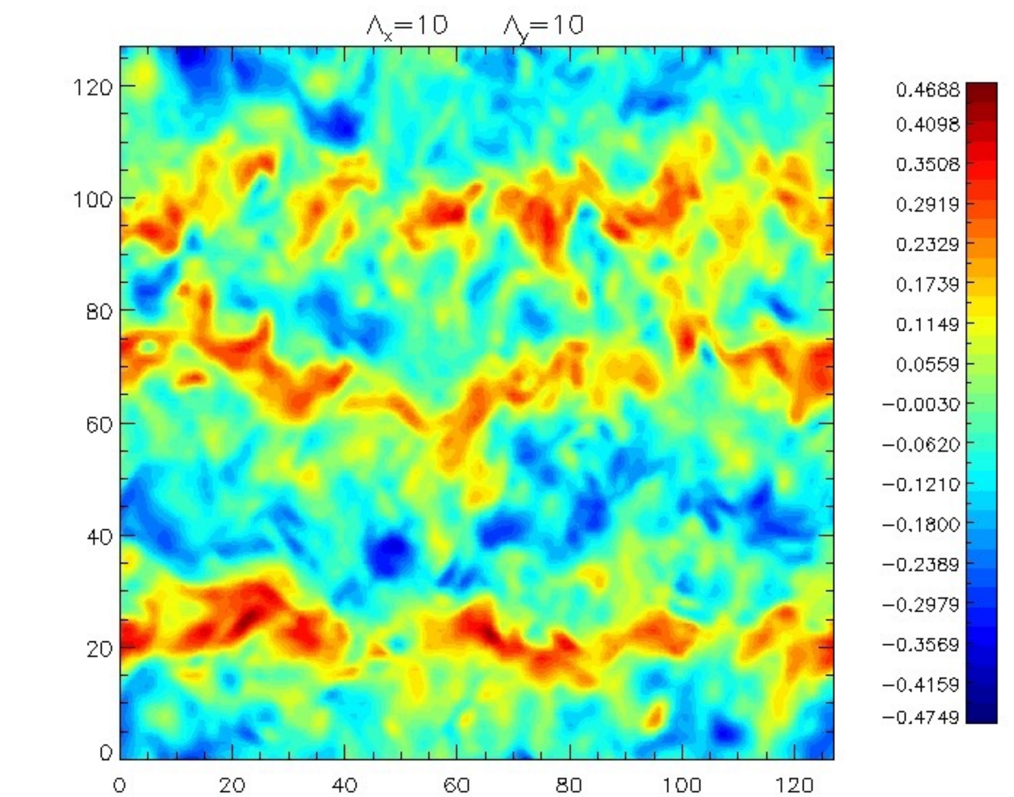

−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
$\Lambda_x=0$ $\Lambda_y=0$
−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
$\Lambda_x=1$ $\Lambda_y=1$
−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
$\Lambda_x=3$ $\Lambda_y=3$
−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
TRUTH
0 5 10 15 20 25

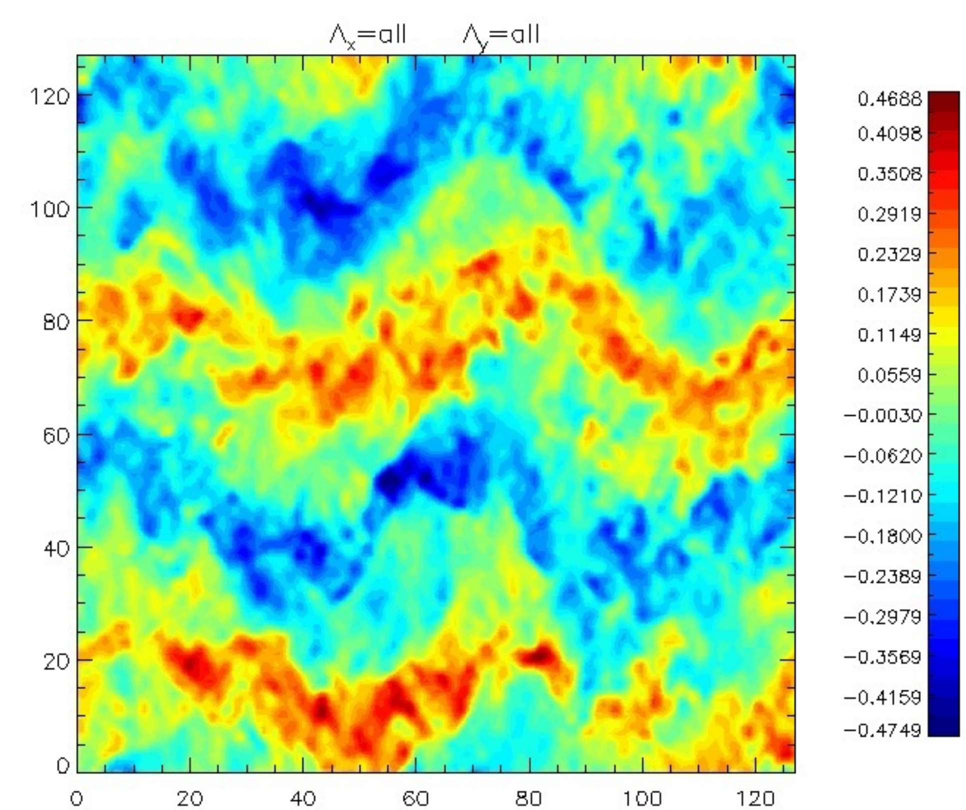

−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
Λ_x=0 Λ_y=64
−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
Λ_x=10 Λ_y=10
−3.59 −3.18 −2.78 −2.38 −1.97 −1.57 −1.16
TRUTH
0 5 10 15 20 25

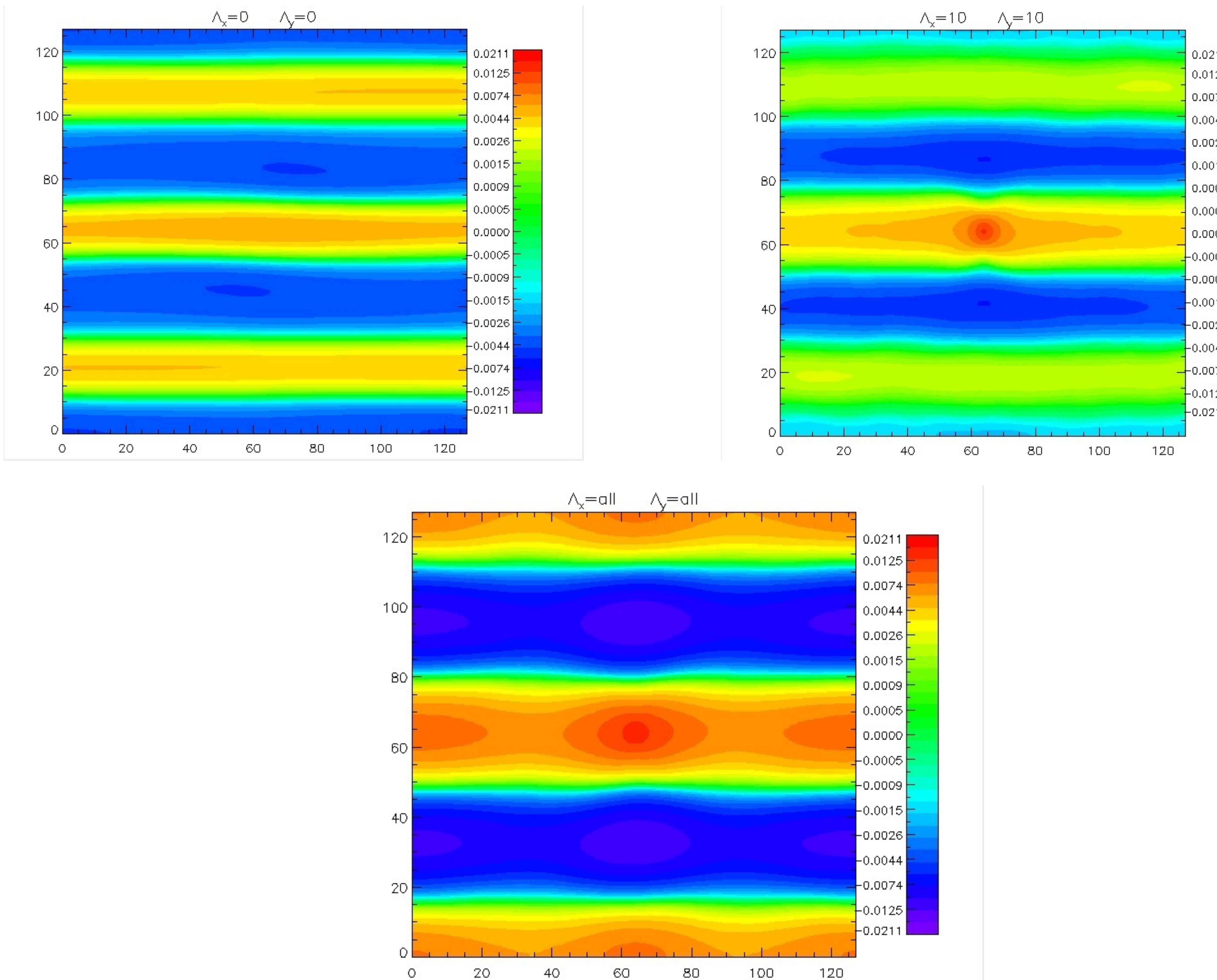

t = 0
u z 0.1
uh z 0.1
v z 0.1
vh z 0.1
x
y


t = 0
u z 0.1
uh z 0.1
v z 0.1
vh z 0.1
x
y


Λx=0 Λy=0
0
20
40
60
80
100
120
0.4688
0.4098
0.3508
0.2919
0.2329
0.1739
0.1149
0.0559
−0.0030
−0.0620
−0.1210
−0.1800
−0.2389
−0.2979
−0.3569
−0.4159
−0.4749


Λx=10 Λy=10
0
20
40
60
80
100
120
0.4688
0.4098
0.3508
0.2919
0.2329
0.1739
0.1149
0.0559
−0.0030
−0.0620
−0.1210
−0.1800
−0.2389
−0.2979
−0.3569
−0.4159
−0.4749


Λx=all Λy=all
0
20
40
60
80
100
120
0.4688
0.4098
0.3508
0.2919
0.2329
0.1739
0.1149
0.0559
−0.0030
−0.0620
−0.1210
−0.1800
−0.2389
−0.2979
−0.3569
−0.4159
−0.4749



Λx=0 Λy=0
Λx=10 Λy=10
Λx=all Λy=all
0 20 40 60 80 100 120
0.0211
0.0125
0.0074
0.0044
0.0026
0.0015
0.0009
0.0005
0.0000
-0.0005
-0.0009
-0.0015
-0.0026
-0.0044
-0.0074
-0.0125
-0.0211

Generalized 2nd Order Cumulant Expansion (GCE2)

$$\frac{\partial}{\partial t} q = L[q] + Q[q,\ q]$$

$$q = \ell + h$$

$$\frac{\partial}{\partial t} \ell = Q[\ell,\ \ell] + Q[(h,\ h)]$$

$$\frac{\partial}{\partial t} h = Q[\ell,\ h]$$

$$\frac{\partial}{\partial t} (h\ h) = 2Q[\ell,\ (h]\ h)$$

Closure

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