Multiscale Equations for Strongly Stratified Turbulent Shear Flows

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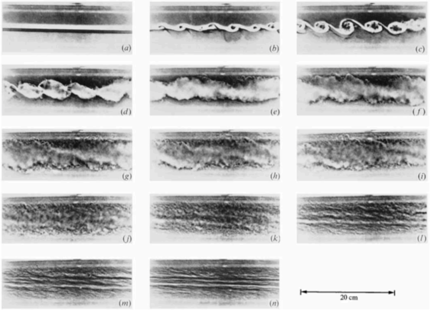
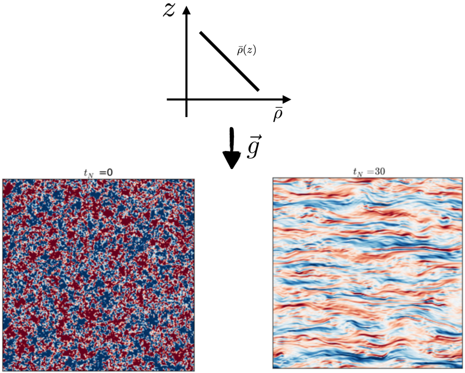
Stratified Turbulence: Phenomenology & Significance



–Thorpe (1971)

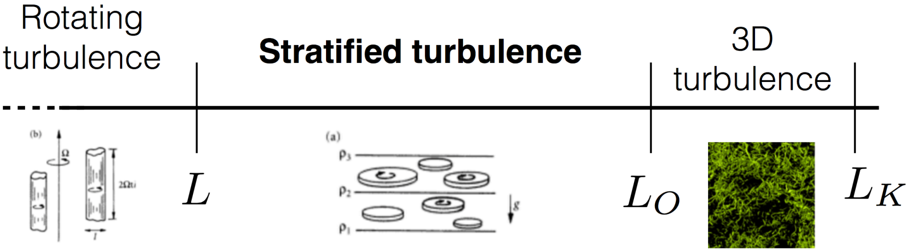


–1024x1024 DNS (C. Rocha)

- Ubiquitous in world's oceans and atmosphere
- Controls diabatic mixing (e.g., crucial for closing ocean circulation)
- SGS process in regional circulation and in computational climate models

Stratified Turbulence: Scales and Parameters

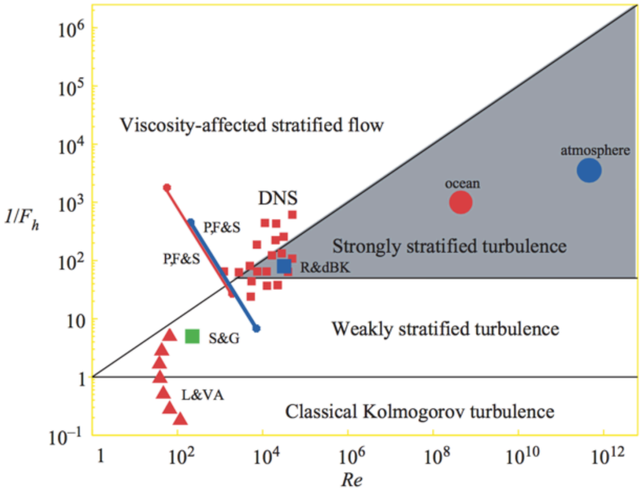
Buoyancy frequency $N \equiv \sqrt{-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}}$



- Oceans: $L \lesssim 10$ km, $L_O \approx 1$ m
- Atmosphere: $L \lesssim 100$ km, $L_O \approx 10$ m
- Reynolds number: $Re \equiv \frac{UL}{\nu}$
- (Horizontal) Froude number: $Fr \equiv \frac{U}{NL}$

Simple scaling arguments give $L/L_O = O(Fr^{-3/2})$, where $L_O = (\epsilon/N^3)^{1/2}$

Stratified Turbulence: Computational Challenge



–Brethouwer *et al.* (2007)

Note: $F_h \equiv Fr$

Stratified Turbulence: Fundamental Questions

- What sets vertical scale?
- Mixing properties?
- Is horizontal spectrum of horizontal KE independent of Fr as $Fr \to 0$ for $Re \gtrsim 10/Fr^2$?

Bartello & Tobias (2013) estimate that to demonstrate this independence even over one decade in Fr, namely, over the parameter range $0.01 \leq Fr \leq 0.1$, would require the ratio of maximum to minimum resolvable scale to be in the **millions** *(i.e. in a single spatial direction), yielding a formidable computational challenge.*

Stratified Turbulence: Governing Equations

Anisotropic Scaling

$\mathbf{x}_\perp : L \qquad z : h \qquad t : L/U \qquad \mathbf{u}_\perp : U \qquad w : Fr^2 UL/h \qquad p : \rho_0 U^2 \qquad b : U^2/h$

Non-Rotating Boussinesq Equations

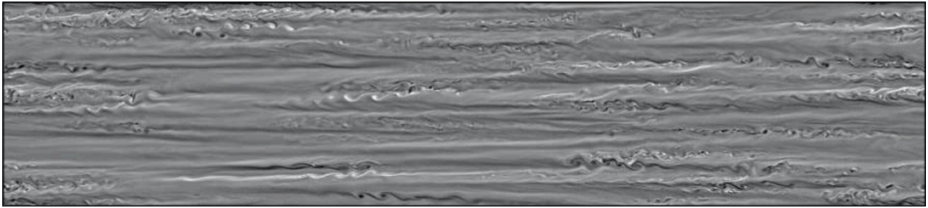
$$\begin{aligned}
\partial_t \mathbf{u}_\perp + (\mathbf{u}_\perp \cdot \nabla_\perp)\mathbf{u}_\perp + \frac{Fr^2}{\alpha^2} W \partial_z \mathbf{u}_\perp &= -\nabla_\perp p + \mathcal{D}(\mathbf{u}_\perp) + \mathbf{f}_\perp \\
Fr^2 \left[\partial_t W + (\mathbf{u}_\perp \cdot \nabla_\perp) W + \frac{Fr^2}{\alpha^2} W \partial_z W \right] &= -\partial_z p + b + Fr^2 \mathcal{D}(W) \\
\nabla_\perp \cdot \mathbf{u}_\perp + \frac{Fr^2}{\alpha^2} \partial_z W &= 0 \\
\partial_t b + (\mathbf{u}_\perp \cdot \nabla_\perp) b + \frac{Fr^2}{\alpha^2} W \partial_z b &= -W + Pr^{-1} \mathcal{D}(b)
\end{aligned}$$

where: $\mathcal{D} = \frac{1}{Re}\left[\nabla_\perp^2 + \frac{1}{\alpha^2}\partial_z^2\right]$ and $\alpha \equiv h/L$, $Fr \equiv U/(NL)$

Limit Equations

- $Fr/\alpha \to 0$ as $Fr \to 0$ (Lilly 1983): Layerwise **2D** flow
- $Fr/\alpha = O(1)$ as $Fr \to 0$ (Billant & Chomaz 2001): Anisotropic **3D** flow

Emergence of Multiple Vertical and Horizontal Scales



–Waite (2014)

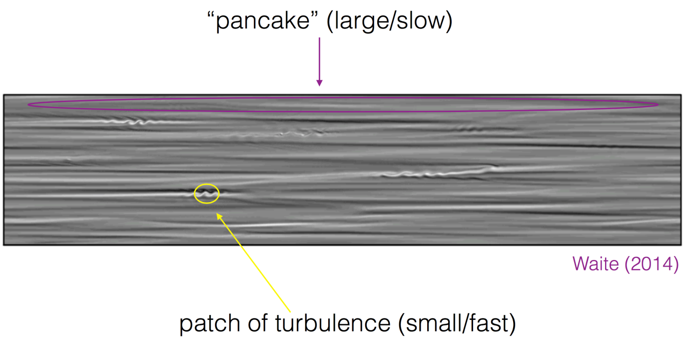
- As $Fr \to 0$, vertical length scale self-adjusts so that $h = O(U/N) \;\Rightarrow\; \alpha = O(Fr)$
- Stratified turbulence regime *defined* by $Fr \to 0$ with:

$\alpha = O(Fr)$ and **buoyancy Reynolds number** $\mathcal{R} \equiv ReFr^2 \gtrsim 10$

- Clear evidence of shear instabilities on horizontal scales $l \ll L$, and perhaps modulation on $O(L)$ vertical scales

. . . *Opportunity for asymptotically-reduced multiscale modeling*. . .

Multiple Scale Asymptotic Analysis



1. Identify relevant **distinguished limit**: $\alpha = Fr$ and $\mathcal{R} \equiv ReFr^2 = O(1)$ as $Fr \to 0$
2. Introduce *fast* horizontal and temporal scales: $\chi_\perp = \mathbf{x}_\perp / Fr$ and $\tau = t/Fr$ so that

$$\nabla_\perp \to \nabla_{\mathbf{x}} + \frac{1}{Fr}\nabla_\chi \qquad \partial_t \to \partial_t + \frac{1}{Fr}\partial_\tau$$

3. Introduce fast averaging operation and mean/fluctuation decomposition:

$$\phi(\mathbf{x}_\perp, z, t) \to \phi(\chi_\perp, \mathbf{x}_\perp, z, \tau, t) = \overline{\phi}(\mathbf{x}_\perp, z, t) + \phi'(\chi_\perp, \mathbf{x}_\perp, z, \tau, t), \text{ where } \overline{\phi'} \equiv 0$$

Expansion *Ansatz*

- Introduce $\epsilon \equiv \sqrt{Fr}$ and posit following asymptotic expansions for various fields:

$$\begin{aligned} [\mathbf{u}_\perp, b, p] &\sim [\mathbf{u}_{0\perp}, b_0, p_0] + \epsilon[\mathbf{u}_{1\perp}, b_1, p_1] + \epsilon^2[\mathbf{u}_{2\perp}, b_2, p_2] + \ldots \\ W &\sim \epsilon^{-1} W_{-1} + W_0 + \epsilon W_1 + \ldots \end{aligned}$$

- Key prescription is that vertical velocity (normalized by αU) is no larger than $O(\epsilon^{-1})$ on fine horizontal scales

- This (re-)scaling ensures that feedback of fluctuations upon mean fields through vertical Reynolds stress divergence $\partial_z \left[\overline{W' u'_\perp}\right]$ arises at proper order:

  Dominant balance with tendency $\partial_t \overline{\mathbf{u}}_\perp$ and vertical diffusion $\mathcal{R}^{-1}\partial_z^2 \overline{\mathbf{u}}_\perp$

  Rescaling simultaneously ensures that fine-scale dynamics are **isotropic**

- Can then *deduce* that **fluctuating** horizontal velocity, buoyancy, and pressure fields arise at $O(\epsilon)$, a key simplification

Multiscale Reduced PDEs

Mean Equations

$$
\begin{aligned}
\left[\partial_t + (\overline{\mathbf{u}}_{0\perp}\cdot\nabla_{\mathbf{x}}) + \overline{W}_0\partial_z\right]\overline{\mathbf{u}}_{0\perp} + \partial_z\left(\overline{W'_{-1}\mathbf{u}'_{1\perp}}\right) &= -\nabla_{\mathbf{x}}\overline{p}_0 + \frac{1}{\mathcal{R}}\partial_z^2\overline{\mathbf{u}}_{0\perp} + \overline{\mathbf{f}}_{0\perp} \\
0 &= -\partial_z\overline{p}_0 + \overline{b}_0 \\
\nabla_{\mathbf{x}}\cdot\overline{\mathbf{u}}_{0\perp} + \partial_z\overline{W}_0 &= 0 \\
\left[\partial_t + (\overline{\mathbf{u}}_{0\perp}\cdot\nabla_{\mathbf{x}}) + \overline{W}_0\partial_z\right]\overline{b}_0 + \partial_z\left(\overline{W'_{-1}b'_1}\right) &= -\overline{W}_0 + \frac{1}{Pr\mathcal{R}}\partial_z^2\overline{b}_0
\end{aligned}
$$

Multiscale Reduced PDEs

Mean Equations

$$\left[\partial_t + (\overline{\mathbf{u}}_{0\perp} \cdot \nabla_{\mathbf{x}}) + \overline{W}_0 \partial_z\right] \overline{\mathbf{u}}_{0\perp} + \partial_z \left(\overline{W'_{-1} \mathbf{u}'_{1\perp}}\right) = -\nabla_{\mathbf{x}} \overline{p}_0 + \frac{1}{\mathcal{R}} \partial_z^2 \overline{\mathbf{u}}_{0\perp} + \overline{\mathbf{f}}_{0\perp}$$

$$0 = -\partial_z \overline{p}_0 + \overline{b}_0$$

$$\nabla_{\mathbf{x}} \cdot \overline{\mathbf{u}}_{0\perp} + \partial_z \overline{W}_0 = 0$$

$$\left[\partial_t + (\overline{\mathbf{u}}_{0\perp} \cdot \nabla_{\mathbf{x}}) + \overline{W}_0 \partial_z\right] \overline{b}_0 + \partial_z \left(\overline{W'_{-1} b'_1}\right) = -\overline{W}_0 + \frac{1}{Pr\mathcal{R}} \partial_z^2 \overline{b}_0$$

Fluctuation Equations

$$(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi)\, \mathbf{u}'_{1\perp} + W'_{-1} \partial_z \overline{\mathbf{u}}_{0\perp} = -\nabla_\chi p'_1 + \frac{Fr}{\mathcal{R}} \left(\nabla_\chi^2 + \partial_z^2\right) \mathbf{u}'_{1\perp}$$

$$(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi)\, W'_{-1} = -\partial_z p'_1 + b'_1 + \frac{Fr}{\mathcal{R}} \left(\nabla_\chi^2 + \partial_z^2\right) b'_1$$

$$\nabla_\chi \cdot \mathbf{u}'_{1\perp} + \partial_z W'_{-1} = 0$$

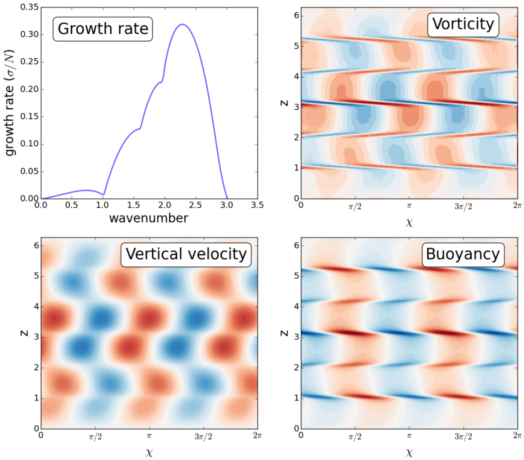
$$(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi)\, b'_1 + W'_{-1} \partial_z \overline{\mathbf{b}}_0 = -W'_{-1} + \frac{Fr}{Pr\mathcal{R}} \left(\nabla_\chi^2 + \partial_z^2\right) b'_1$$

Attributes of Multiscale Equations

- In absence of Reynolds stress ("eddy-flux") divergences (RSDs), mean equations reduce to hydrostatic primitive equations (Billant & Chomaz 2001)
- Vertical RSDs provide crucial feedback of fluctuating fine-scale dynamics on evolution of mean fields, given here without need for *ad hoc* closure
- Fluctuation dynamics are **quasi-linear (QL)** about local mean fields:
  - Reverting to single of $(\mathbf{x}_\perp, t)$ scales yields [and interpreting $\overline{(\cdot)}$ as strict horizontal mean] yields a **QL reduction** of full Boussinesq equations
  - Suggests **2nd-order cumulant expansion (CE2**) approaches used by Marston & Tobias, Farrell & Ioannou, and Young & Srinivasan can be **formally justified** for stratified shear turbulence in the limit $Fr \to 0$
- By retaining multiple horizontal and temporal scales, can rationally extend popular QL/CE2 schemes $\Rightarrow$ Spectral space interpretation has led to considerably more accurate **GQL/GCE2** formulation (Marston, Tobias & Chini 2015)

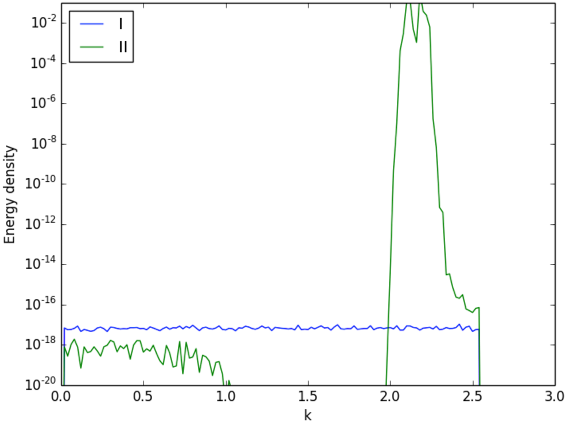
Preliminary QL Results for Sinusoidally-Forced 2D Flow: $\overline{f}_{0x} = (m^2/\mathcal{R})\sin(mz)$

Linear Stability of Laminar Mean State ($m = 3,\ \mathcal{R} = 200,\ Fr = 0.02$)



Preliminary QL Results for 2D Flow: Evolution of Fluctuation and Mean Fields

Preliminary QL Results for 2D Flow: Evolution of Horizontal KE Spectrum



$\Rightarrow$ Evolution toward narrow banded spectrum

Extension of Model: Beyond QL Dynamics

- As $Fr \to 0$, leading-order fluctuation equations are **inviscid/non-dissipative**, and satisfy Taylor–Goldstein (TG) equation in 2D

$$\frac{d^2\hat{\psi}}{dz^2} + \left[\frac{\left(1 + \partial_z \overline{b}_0\right)}{\left(\overline{u}_0 - c\right)^2} - \frac{\partial_z^2 \overline{u}_0}{\left(\overline{u}_0 - c\right)} - \alpha^2 \right] \hat{\psi} \quad = \quad 0$$

  for normal modes $\psi'(\chi, z, \tau) = \hat{\psi}(z) e^{i\alpha(\chi - c\tau)} + c.c.$, where $c \equiv \omega/\alpha$ and $(u_1', W_{-1}) \equiv (\partial_z \psi', -\partial_\chi \psi')$

- Regular unstable modes, but singular neutral modes (with discontinuous first derivatives) and potentially singular marginal modes

- Self-organized criticality? DNS of stratified shear layer (Werne & Fritts 1999) indicates gradient Richardson no. $Ri_g \equiv \left(1 + \partial_z \overline{b}_0\right) / ||\partial_z \overline{u}_0||_\infty^2 \to 1/4$ for late time

- These considerations suggest the potential emergence of dynamics on *finer* z-scale – in fact, on the Ozmidov scale. . .

Extension of Model: Scaling Arguments

- In nbhd. of z_0 level, where vertical gradients may be large, diffusion important in fluctuation dynamics when thickness of layer is s.t.

$$\boxed{\frac{\epsilon^2}{\mathcal{R}}\partial_z^2 \mathbf{u}'_{1\perp} \quad = \quad O\left(W'_{-1}\partial_z \overline{u}_{0\perp}\right) \quad \Rightarrow \quad \text{i.e. when } \partial_z = O(\epsilon^{-1})}$$

- Let $z = z_0 + \epsilon\eta$ (recalling $\epsilon \equiv \sqrt{Fr}$) and introduce "inner" fields/expansions:

$$\begin{aligned}
\mathbf{u}_\perp &= \mathbf{U}_\perp(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t; \epsilon) \sim \overline{\mathbf{u}}_{0\perp}(\mathbf{x}_\perp, z_0, t) + \epsilon \mathbf{U}_{1\perp}(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t) + \ldots \\
W &= \mathcal{W}(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t; \epsilon) \sim W'_{-1}(\chi_\perp, \mathbf{x}_\perp, z_0, \tau, t) + \mathcal{W}_0(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t) + \ldots \\
p &= P(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t; \epsilon) \sim \overline{p}_0(\mathbf{x}_\perp, z_0, t) + \epsilon p_1(\chi_\perp, \mathbf{x}_\perp, z_0, \tau, t) + \ldots \\
b &= B(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t; \epsilon) \sim \overline{b}(\mathbf{x}_\perp, z_0, t) + \epsilon B_1(\chi_\perp, \mathbf{x}_\perp, \eta, \tau, t) + \ldots
\end{aligned}$$

Extended Model

Corrections to Slope of Mean Fields

$$\frac{1}{\mathcal{R}}\partial_\eta^2\overline{\mathbf{U}}_{1\perp} = \partial_\eta\left(\overline{W'_{-1}\mathbf{U}'_{1\perp}}\right) \Rightarrow \frac{1}{\mathcal{R}}\partial_\eta\overline{\mathbf{U}}_{1\perp} = W'_{-1}\left(\mathbf{U}'_{1\perp} - \mathbf{U}'_{1\perp}|_{z_0}\right)$$

$$\frac{1}{Pr\mathcal{R}}\partial_\eta^2\overline{B}_1 = \partial_\eta\left(\overline{W'_{-1}B'_1}\right) \Rightarrow \frac{1}{Pr\mathcal{R}}\partial_\eta\overline{B}_{1\perp} = W'_{-1}\left(B'_1 - B'_1|_{z_0}\right)$$

Nonlinear Fluctuation Dynamics

$$\left[\partial_\tau + \overline{\mathbf{u}}_{0\perp}\cdot\nabla_\chi\right]\mathbf{U}'_{1\perp} + W'_{-1}\partial_\eta\left(\overline{\mathbf{U}}_{1\perp} + \mathbf{U}'_{1\perp}\right) - \partial_\eta\left(\overline{W'_{-1}\mathbf{U}'_{1\perp}}\right) = -\nabla_\chi p'_1 + \frac{1}{\mathcal{R}}\partial_\eta^2\mathbf{U}'_{1\perp}$$

$$\left[\partial_\tau + \overline{\mathbf{u}}_{0\perp}\cdot\nabla_\chi\right]B'_1 + W'_{-1}\partial_\eta\left(\overline{B}_1 + B'_1\right) - \partial_\eta\left(\overline{W'_{-1}B'_1}\right) = -W'_{-1} + \frac{1}{Pr\mathcal{R}}\partial_\eta^2 B'_1$$

Observation: Fluctuation equations bear certain similarities to reduced PDEs derived by Balmforth & Young (1997) for dynamics of shear flows with "vorticity defects".

Summary

- Separation of scales, strong anisotropies emergent in extreme parameter regimes present obstacles for DNS but opportunities for reduced multiscale modeling
- Certain QL/CE2 reductions may be formally justified via asymptotic analysis, particularly for flows subjected to strong restraints (strong stratification)
- Asymptotic analysis also suggests important ways to extend QL/CE2 models:
  - Slow variation of mean fields
  - Sub-regions of flow in which fluctuation nonlinearities may be non-negligible
- Future directions:
  - Implement multiscale numerical scheme to incorporate slow modulation of pancake structures
  - Pursue statistical implementation (e.g. CE2)
  - Incorporate "inner" layer dynamics on (vertical) Ozmidov scale