Reduced description of exact coherent states in parallel shear flows

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Asymptotic reduction of nonlinear flows

Basic idea:

Strong constraint $\Rightarrow$ reduce the flow in a particular direction, anisotropy

$\Downarrow$

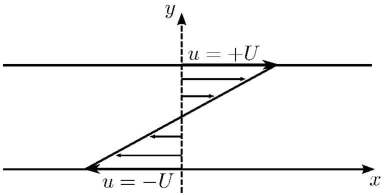
Small parameter $\Rightarrow$ asymptotically consistent simplification of equations

Shear flow is a natural candidate for this type of approach

- Boundary layers: P. Hall & W. D. Lakin, *Proc. R. Soc. London A* (1988)
- Rayleigh–Bénard convection: P. J. Blennerhassett & A. P. Bassom, *IMA J. Appl. Math.* (1994)
- Strongly constrained convection: K. Julien & E. Knobloch, *J. Math. Phys.* (2007)
- Langmuir circulation: G. P. Chini, K. Julien & E. Knobloch *Geophys. Astrophys. Fluid Dyn.* (2009)

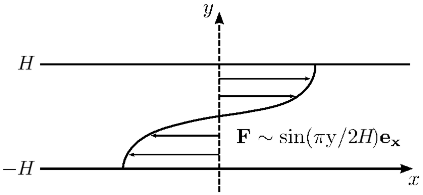
Plane parallel shear flows

Plane Couette Flow



Wall BCs: $u = \pm 1,\ v = w = 0$
Forcing: $\mathbf{f}(y) = \mathbf{0}$

Waleffe Flow



Wall BCs: $\partial_y u = 0,\ v = 0,\ \partial_y w = 0$
Forcing: $\mathbf{f}(y) = \frac{\sqrt{2}\pi^2}{4Re} \sin\left(\frac{\pi y}{2}\right)\hat{\mathbf{e}}_x$

Waleffe, *Phys. Fluids* **9** 883–900 (1997)

Incompressible Navier–Stokes equations

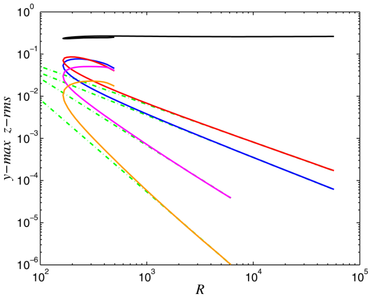
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{v} + \mathbf{f}$$

$$\nabla\cdot\mathbf{v} = 0 \qquad \text{with} \quad Re \equiv UH/\nu$$

Asymptotic scaling

Basic knowledge: **streamwise rolls** and **fluctuations** are *weak* compared to **streamwise streaks**; self-sustaining process

Key paper: Lower branch states in plane Couette flow:



Fourier decomposition for steady-state ECS:

$$\boxed{\mathbf{v}(\mathbf{x}) = \sum_{n=-\infty}^{n=+\infty} \hat{\mathbf{v}}_n(y,z)e^{in\alpha x}}$$

Scalings:

- $\hat{u}_0 = O(1)$
- $(\hat{v}_0, \hat{w}_0) = O(Re^{-1})$
- $\hat{\mathbf{v}}_1 = O(Re^{-0.9})$
- $\hat{\mathbf{v}}_n = o(Re^{-1})$ for $n > 1$

Wang *et al.*, *Phys. Rev. Lett.* **98** 204501 (2007)

Multiscale analysis

- Define $\epsilon \equiv 1/Re \ll 1$
- $X = \epsilon x \Rightarrow \partial_x \to \partial_x + \epsilon \partial_X$

  $T = \epsilon t \Rightarrow \partial_t \to \partial_t + \epsilon \partial_T$
- Decompose: $(\mathbf{v}, p) = (\bar{\mathbf{v}}, \bar{p})(X, y, z, T) + (\mathbf{v}', p')(x, X, y, z, t, T)$
  $\overline{(\cdot)}$ = average over (x,t), and $(\cdot)'$ = fluctuation about mean
- Define $\mathbf{v} = u\hat{\mathbf{e}}_\mathbf{x} + \mathbf{v}_\perp$ and expand, following Wang *et al* (2007):

$$\boxed{\begin{aligned} u &\sim \bar{u}_0 + u_0' + \epsilon\left(\bar{u}_1 + u_1'\right) + \dots \\ \mathbf{v}_\perp &\sim \epsilon\left(\bar{\mathbf{v}}_{1\perp} + \mathbf{v}_{1\perp}'\right) + \dots \\ p &\sim \bar{p}_0 + p_0' + \epsilon\left(\bar{p}_1 + p_1'\right) + \epsilon^2\left(\bar{p}_2 + p_2'\right) + \dots \end{aligned}}$$

Note: since at $\mathcal{O}(1)$ we have $\partial_x u_0' = 0$ it follows that $u_0' = p_0' \equiv 0$.

Leading order

At $\mathcal{O}(\epsilon)$

$$\begin{aligned}\partial_t \bar{u}_1' \quad &+ \quad \partial_T \bar{u}_0 + \bar{u}_0 \partial_X \bar{u}_1' + \bar{u}_0 \partial_X \bar{u}_0 + \left[\left(\bar{\mathbf{v}}_{1\perp} + \mathbf{v}_{1\perp}' \right) \cdot \nabla_\perp \right] \bar{u}_0 \\ &= \quad -\partial_x p_1' - \partial_X \bar{p}_0 + \nabla_\perp^2 \bar{u}_0 + f(y)\end{aligned}$$

Averaging this equation over fast x and fast t:

$$\partial_T \bar{u}_0 + \bar{u}_0 \partial_X \bar{u}_0 + \left(\bar{\mathbf{v}}_{1\perp} \cdot \nabla_\perp \right) \bar{u}_0 = -\partial_X \bar{p}_0 + \nabla_\perp^2 \bar{u}_0 + f(y)$$

Subtracting the second equation from the first:

$$\partial_t \bar{u}_1' + \bar{u}_0 \partial_x \bar{u}_1' + \left(\mathbf{v}_{1\perp}' \cdot \nabla_\perp \right) \bar{u}_0 = -\partial_x p_1'$$

Similarly:

$$\partial_t \mathbf{v}_{1\perp}' + \bar{u}_0 \partial_x \mathbf{v}_{1\perp}' = -\nabla_\perp \left(\bar{p}_1 + p_1' \right)$$

Thus $\nabla_\perp \bar{p}_1 = 0$. Finally, from the continuity equation we obtain

$$\partial_X \bar{u}_0 + \nabla_\perp \cdot \bar{\mathbf{v}}_{1\perp} = 0, \qquad \partial_x u_1' + \nabla_\perp \cdot \mathbf{v}_{1\perp}' = 0.$$

Resulting equations

To obtain a closed system we average the $\mathcal{O}(\epsilon^2)$ perpendicular momentum equation:

$$\partial_T \bar{\mathbf{v}}_{1\perp} + \partial_X \left[\bar{u}_0 \bar{\mathbf{v}}_{1\perp}\right] + \nabla_\perp \cdot \left[\bar{\mathbf{v}}_{1\perp}\bar{\mathbf{v}}_{1\perp} + \overline{\mathbf{v}'_{1\perp}\mathbf{v}'_{1\perp}}\right] = \nabla_\perp \bar{p}_1 + \nabla^2_\perp \bar{\mathbf{v}}_{1\perp}$$

When $\partial_X \equiv 0$, and $\mathbf{v_1}'(x,y,z,t,T) = \mathbf{v_1}(y,z,t,T)e^{i\alpha x} + c.c.$ we have:

Mean equations

$$\begin{aligned} \partial_T \bar{u}_0 + \left(\bar{\mathbf{v}}_{1\perp} \cdot \nabla_\perp\right) \bar{u}_0 &= \nabla^2_\perp \bar{u}_0 + f(y) \\ \partial_T \bar{\mathbf{v}}_{1\perp} + \nabla_\perp \cdot \left[\bar{\mathbf{v}}_{1\perp}\bar{\mathbf{v}}_{1\perp} + \overline{\mathbf{v}'_{1\perp}\mathbf{v}'_{1\perp}}\right] &= -\nabla_\perp \bar{p}_2 + \nabla^2_\perp \bar{\mathbf{v}}_{1\perp} \\ \nabla_\perp \cdot \bar{\mathbf{v}}_{1\perp} &= 0 \end{aligned}$$

Fluctuation equations

$$\begin{aligned} \partial_t u'_1 + \bar{u}_0 i\alpha u'_1 + \left(\mathbf{v}'_{1\perp} \cdot \nabla_\perp\right) \bar{u}_0 &= -i\alpha p'_1 \\ \partial_t \mathbf{v}'_{1\perp} + \bar{u}_0 i\alpha \mathbf{v}'_{1\perp} &= -\nabla_\perp p'_1 \\ i\alpha u'_1 + \nabla_\perp \cdot \mathbf{v}'_{1\perp} &= 0 \end{aligned}$$

Reduced system

Streamfunction-vorticity: $\bar{v}_1 = -\partial_z \phi_1, \quad \bar{w}_1 = \partial_y \phi_1, \quad \omega_1 = \nabla_\perp^2 \phi_1$

Mean equations

$$
\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) \\
\partial_T \omega_1 + J(\phi_1, \omega_1) &+ 2(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*)) \\
&+ 2\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*) = \nabla_\perp^2 \omega_1
\end{aligned}
$$

$J(a, b) = \partial_y a \partial_z b - \partial_z a \partial_y b$, $\quad \mathcal{R}$ real part, $\quad ^*$ complex conjugate

Reduced system

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$J(a,b) = \partial_y a \partial_z b - \partial_z a \partial_y b,$ $\quad \mathcal{R}$ real part, $\quad ^*$ complex conjugate

Eliminating pressure:

Fluctuation equations

$$
\begin{aligned}
(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) \\
\partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1
\end{aligned}
$$

Reduced system

Reduced system

$$\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) & (1)\\
\partial_T \omega_1 + J(\phi_1, \omega_1) &= \nabla_\perp^2 \omega_1 - 2(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*)) - 2\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*) & (2)\\
(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) & (3)\\
\partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1 + \epsilon \nabla_\perp^2 \mathbf{v}_{1\perp} & (4)
\end{aligned}$$

$J(a,b) = \partial_y a \partial_z b - \partial_z a \partial_y b$, $\quad \mathcal{R}$ real part, $\quad {}^*$ complex conjugate

- 2D system (y, z) but 3 components (streamwise, wall-normal, spanwise)
- Mean system (1)–(2) has unit effective Re
- Fluctuation equations (3)–(4) are: **inviscid**; quasi-linear and singular

$$(\alpha^2 - \nabla_\perp^2) p_1 + \frac{2}{u_0}\left(\nabla_\perp u_0 \cdot \nabla_\perp p_1 - \epsilon \nabla_\perp u_0 \cdot \nabla_\perp^2 \mathbf{v}_{1\perp}\right) = 0$$

Generalized Rayleigh equation

- **Critical regions!**

Iterative algorithm

Reduced model

$$
\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) && (5)\\
\partial_T \omega_1 + J(\phi_1, \omega_1) &= \nabla_\perp^2 \omega_1 - 2(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*)) - 2\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*) && (6)\\
(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) && (7)\\
\partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1 + \epsilon \nabla_\perp^2 \mathbf{v}_{1\perp} && (8)
\end{aligned}
$$

Step 1: choose a fluctuation amplitude A and a profile u_0

Step 2: compute the fastest non-oscillatory growing $\mathbf{v}_{1\perp}$ mode

Step 3: use A and the result of Step 2 to compute the Reynolds stresses

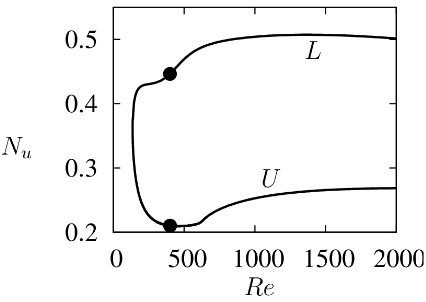
Step 4: time-advance u_0 and ω_1 to a steady state

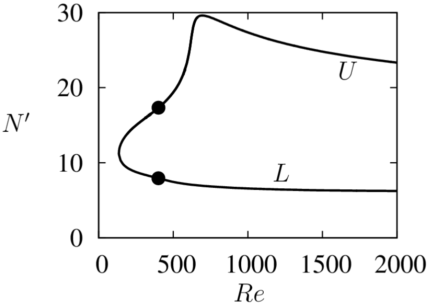
Then: repeat Steps 2–4 until convergence

Repeat for different A to find A_{opt} for which the converged solution has marginal fluctuations (i.e. $\partial_t \mathbf{v}_{1\perp} = 0$).

Results for Waleffe flow: $\alpha = 0.5$, $L_z = \pi$

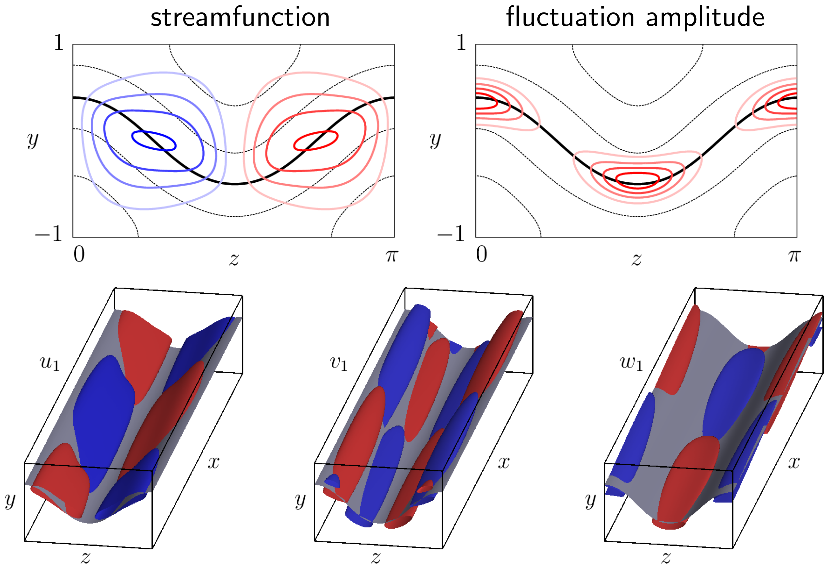
$N_u \equiv \mathcal{D}^{-1} \int_{\mathcal{D}} u_0^2 \, dy \, dz$ $\qquad$ $N' \equiv \mathcal{D}^{-1} \int_{\mathcal{D}} (v_1^2 + w_1^2) \, dy \, dz$



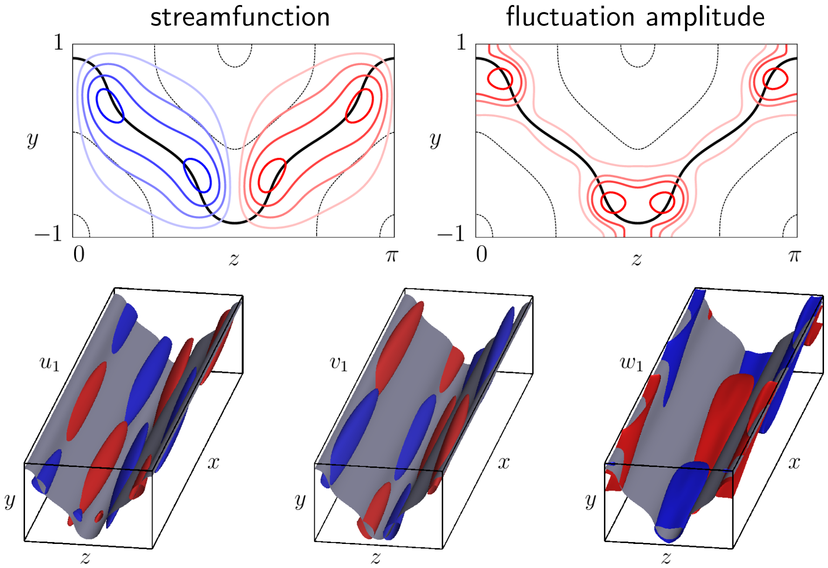


Note that trivial solution has $N_u = 1$ and $N' = 0$.

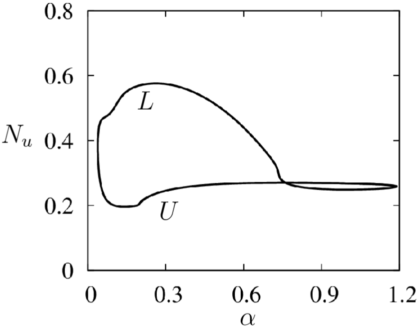
Lower branch states: $Re = 1500$, $\alpha = 0.5$, $L_z = \pi$

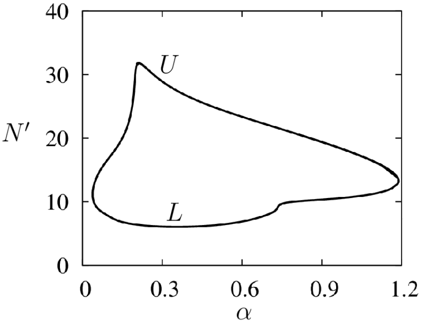


Upper branch states: $Re = 1500$, $\alpha = 0.5$, $L_z = \pi$



Dependence on α: $Re = 1500$, $L_z = \pi$

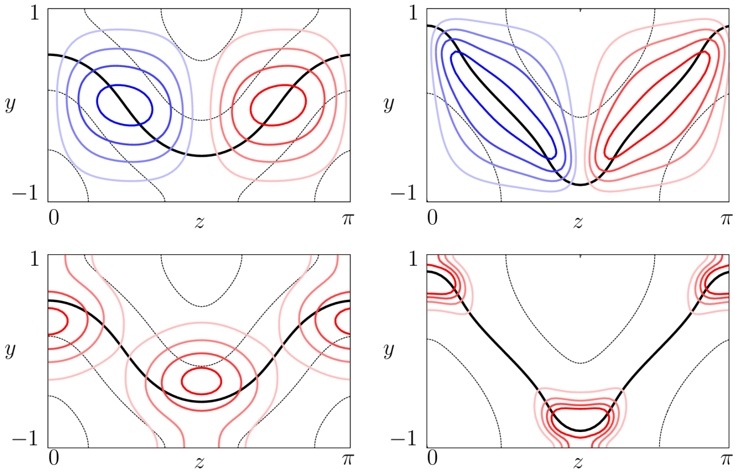




- Isola when continued versus streamwise period $L_x = 2\pi/\alpha$
- ECS exist down to $\mathcal{O}(1)$ streamwise periods: $\alpha \approx 1.1890 \Rightarrow L_x \approx 5.3$
- ECS exist up to large streamwise periods: $\alpha \approx 0.0380 \Rightarrow L_x \approx 165$

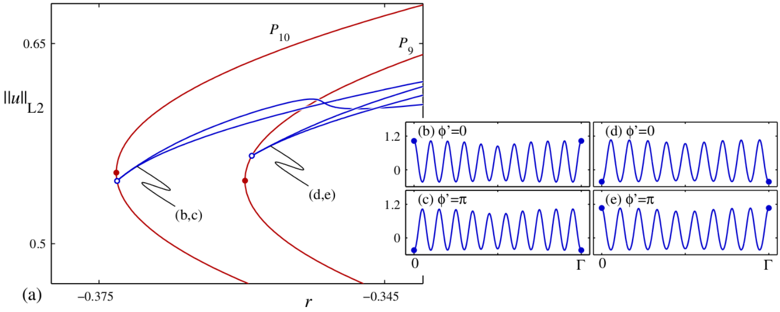
ECS of extreme streamwise period: $Re = 1500,\ L_z = \pi$

$\alpha \approx 0.0380 \Rightarrow L_x \approx 165$ $\alpha \approx 1.1890 \Rightarrow L_x \approx 5.3$



Modulated patterns: The postulate

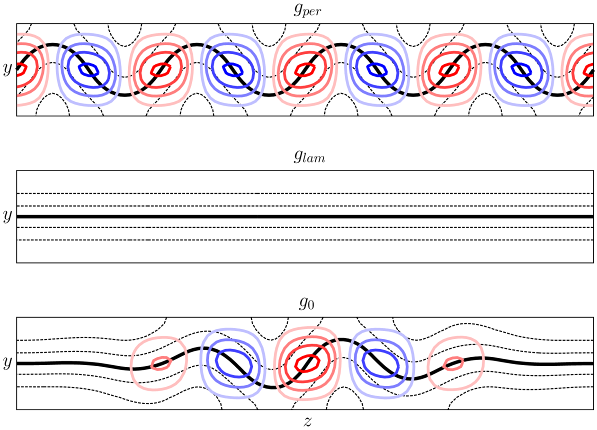
Saddle-nodes of subcritical branches in large domains yield modulational instabilities



Bergeon *et al.*, *Phys. Rev. E* (2008)

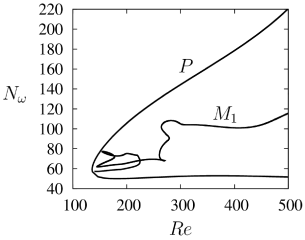
Modulated patterns: Artificial modulation

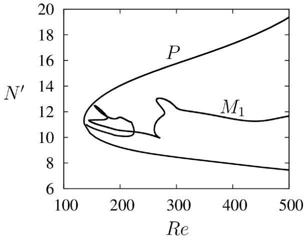
Extend solutions to a $L_z = 4\pi$ domain



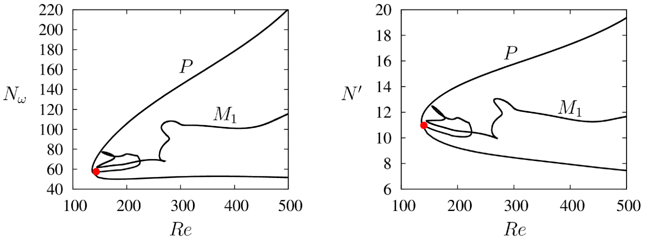
$$g_0 = \left[1 - \frac{\chi}{2}\left(1 + \cos\left(\frac{z}{2}\right)\right)\right] g_{per} + \left[\frac{\chi}{2}\left(1 + \cos\left(\frac{z}{2}\right)\right)\right] g_{lam}$$

Modulated patterns: M_1 states, $L_z = 4\pi$

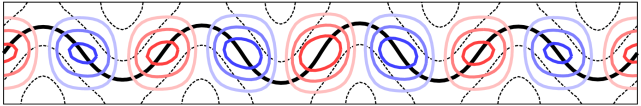




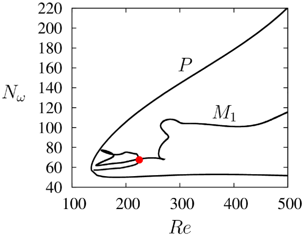
Modulated patterns: M_1 states, $L_z = 4\pi$

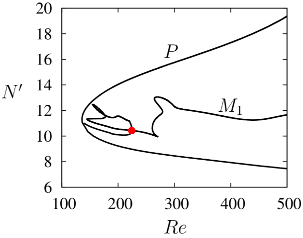


$Re \approx 140$

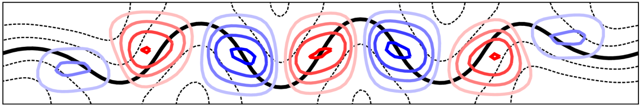


Modulated patterns: M_1 states, $L_z = 4\pi$

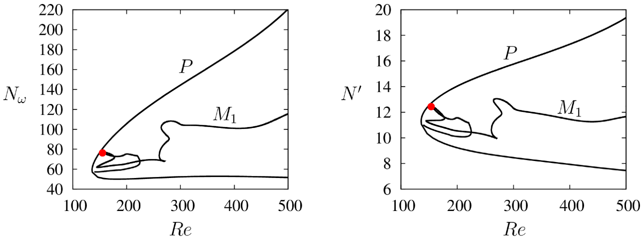




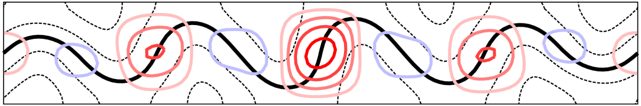
$Re \approx 225$



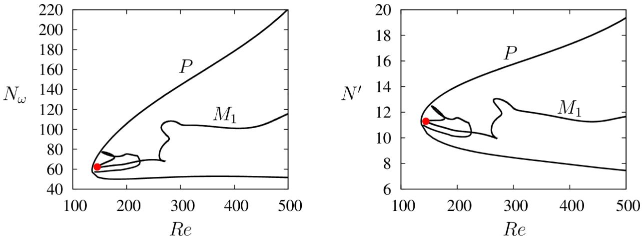
Modulated patterns: M_1 states, $L_z = 4\pi$



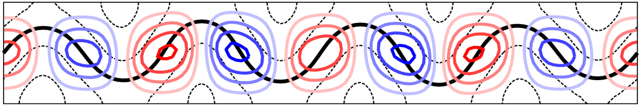
$Re \approx 155$



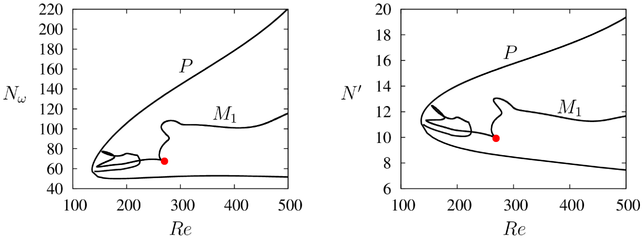
Modulated patterns: M_1 states, $L_z = 4\pi$



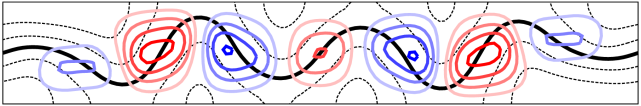
$Re \approx 145$



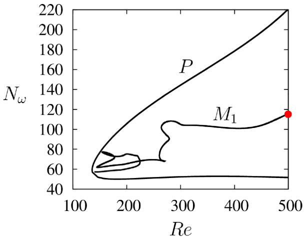
Modulated patterns: M_1 states, $L_z = 4\pi$

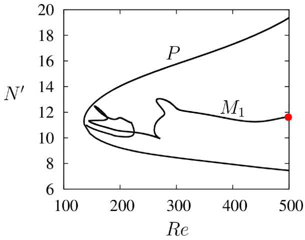


$Re \approx 271$

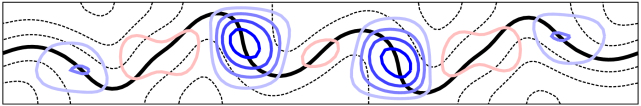


Modulated patterns: M_1 states, $L_z = 4\pi$





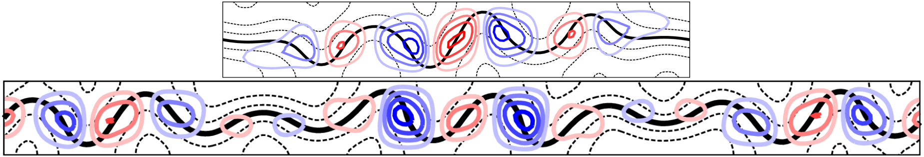
$Re \approx 500$



Conclusions

- Closed reduced description of ECS in parallel shear flows
- Regularization by subdominant dissipation $\Rightarrow$ critical region dynamics
- Efficient novel numerical technique
- Lower, upper branches and modulated states computed

An attempt at large domains:



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- Beaume, Knobloch, Chini and Julien, *Fluid Dyn. Res.* **47**, 015504 (2015)
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- Beaume, Knobloch, Chini and Julien, *On a reviewer's desk* (2015)