Structure and Mechanism in Second Order Statistical State Dynamics Models of Wall-bounded Shear Flow Turbulence

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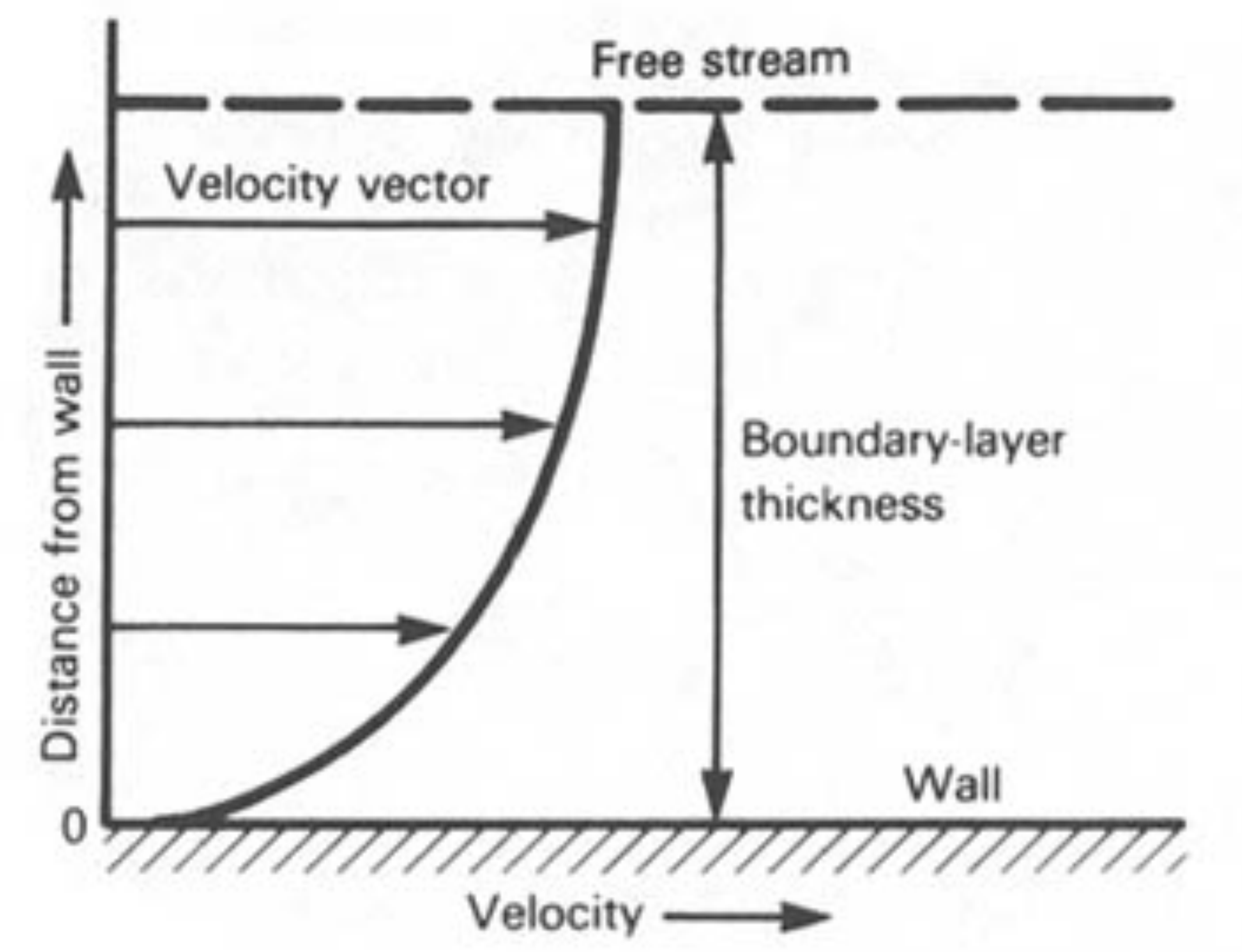
EUNH/IAM Workshop on Advancing Wall-Turbulence Model Development and Implementation

November 19-20, 2015

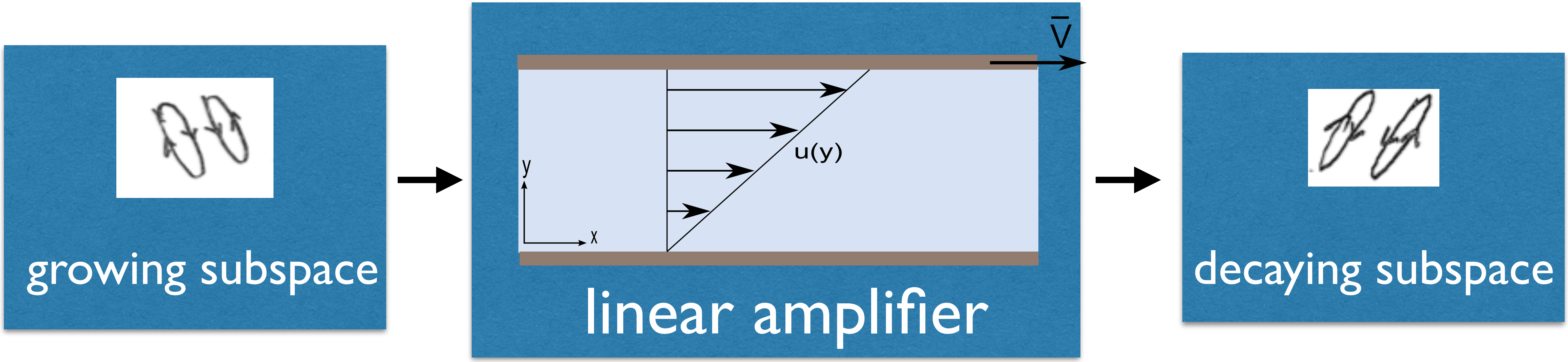
thanks to: Petros Ioannou, Dennice Gayme, Vaughan Thomas, Javier Jimenez, Binh Lieu, Mihailo Jovanovic, Nikos Constantinou, Adrian Lozano-Duran, Marios Nikolaidis

- Observation and simulation of wall-turbulence has greatly advanced while understanding of the mechanisms underlying turbulence remains incomplete.
- Linear theory provides insight into the dynamics of turbulence not available from direct inspection of observations and simulations of the turbulent state.
- While turbulence is not sustained in linear theory, the simplest extension of linear theory, quasi-linear theory, sustains realistic turbulence and, moreover, can be completely characterized.
- While QL SSD models can’t replace DNS for the purpose of simulation, QL/SSD theory provides a comprehensive understanding of the physical mechanism underlying wall-turbulence.

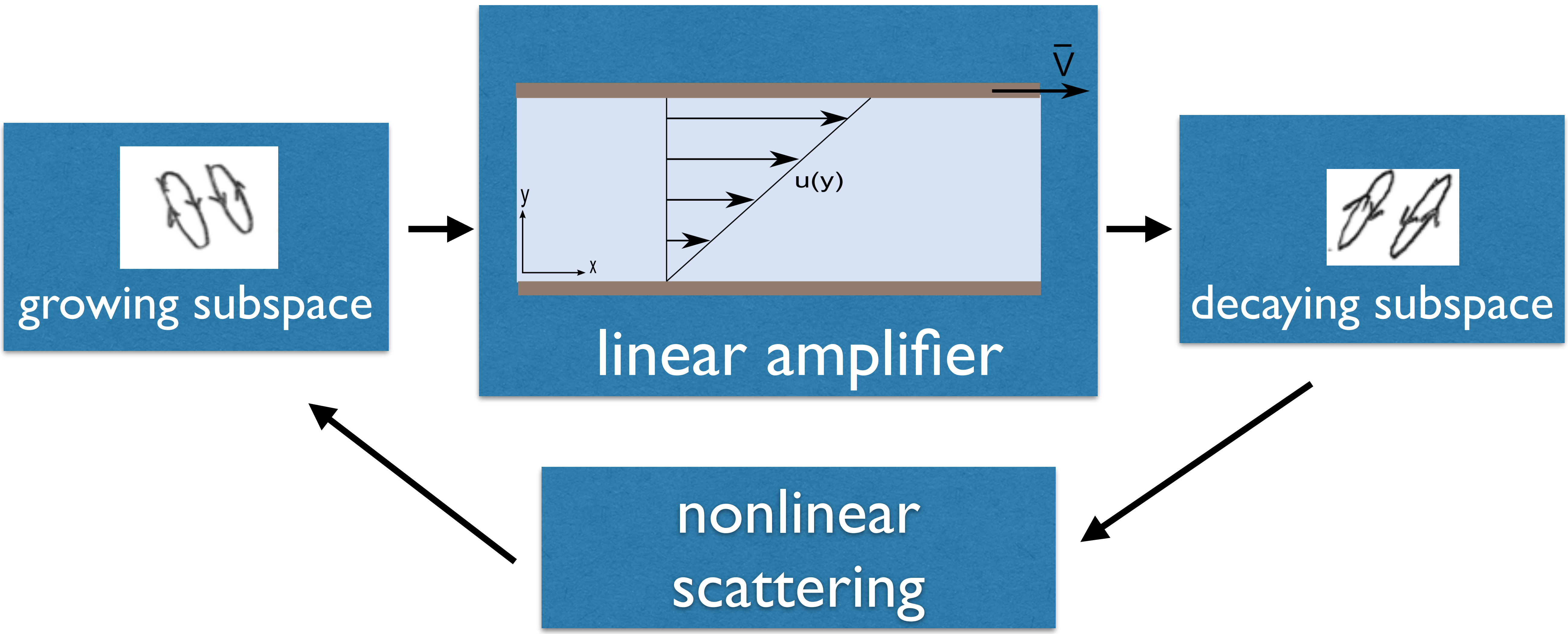
Wall-turbulence in devices and atmospheres is fundamentally linear



Turbulence Theory (linear energetics)



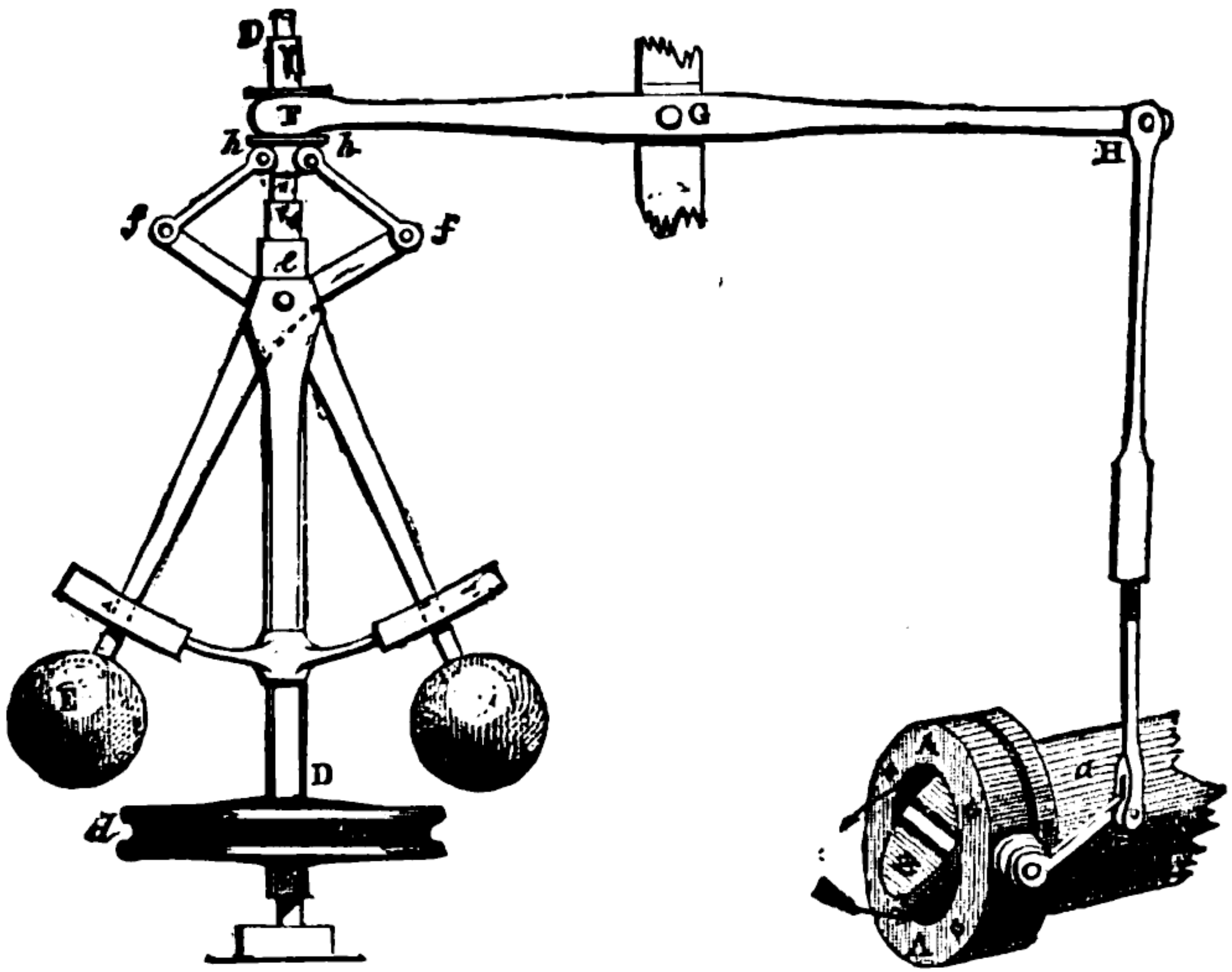
Turbulence Theory (nonlinear feedback)



This nonlinear feedback process instigates a runaway increase in turbulence intensity.

Turbulence Theory (the governor)

This divergence must be limited by some mechanism in order to maintain a statistically steady turbulent equilibrium.

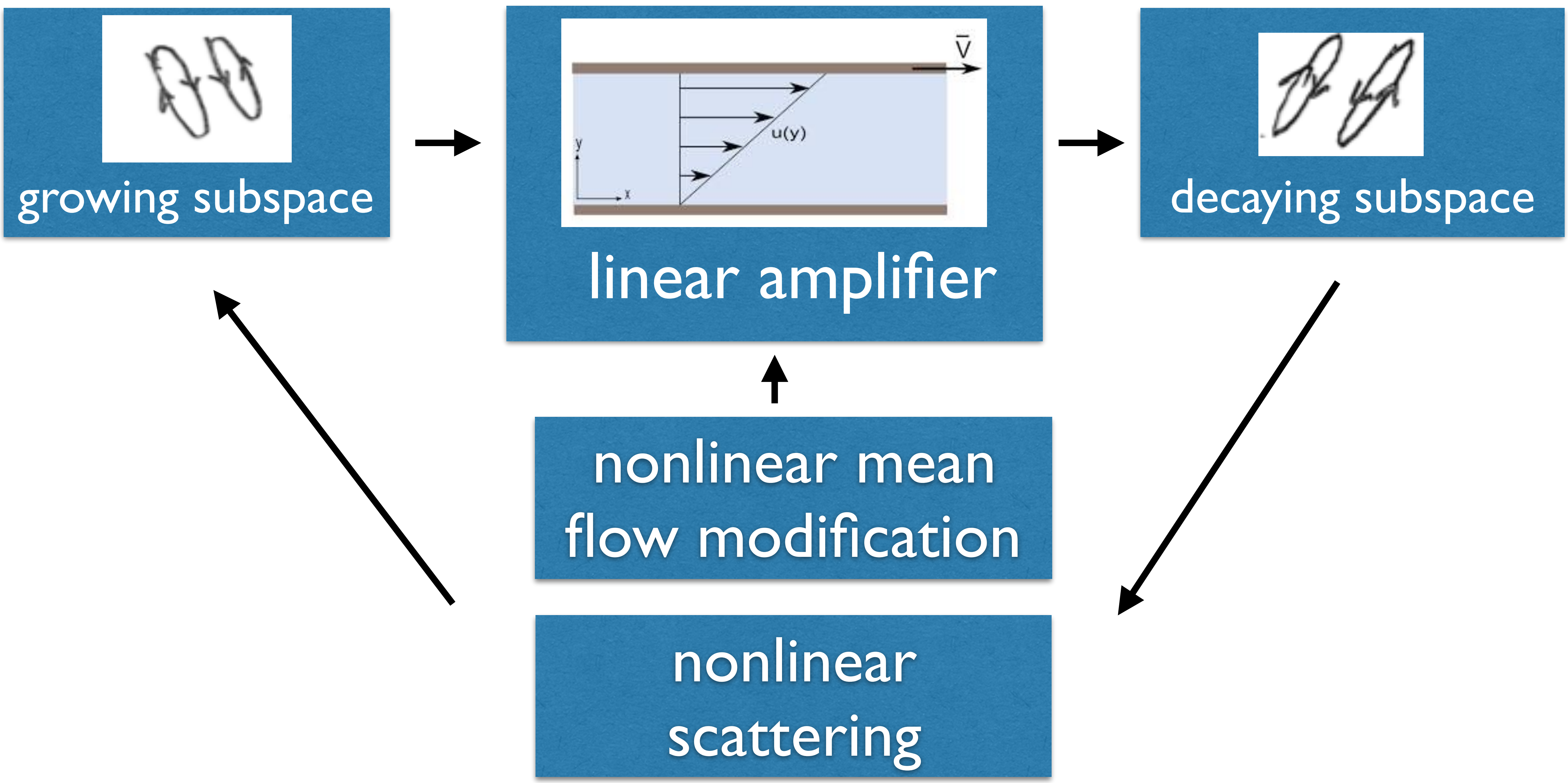


James Watt 1788

Turbulence Theory (synthesis)

Divergence is intercepted by a nonlinear perturbation-mediated mean flow modification that acts as a feedback control enforcing the statistically steady observed turbulent equilibrium.

This control has analytical expression only in SSD



Navier-Stokes Equations in mean/perturbation form

$$\bar{\mathbf{u}}_{tot} = \mathbf{U} + \mathbf{u} = (U, V, W) + (u, v, w)$$

$\mathbf{U}$: Streamwise mean velocity
$\mathbf{u}$: Perturbation velocity

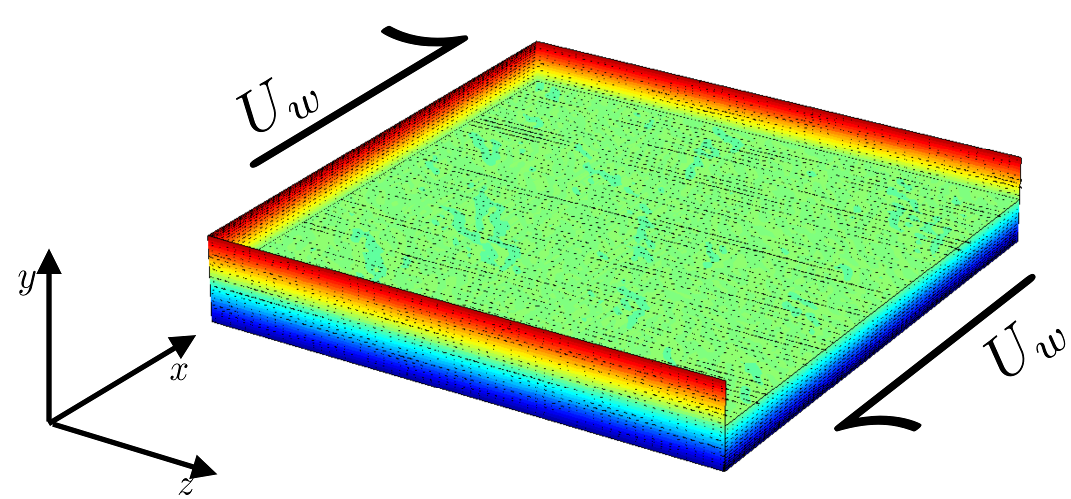
2δ : channel height
$\pm U_w$: wall velocities

$$\mathrm{R} = \frac{U_w \delta}{\nu}$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U} = - < \mathbf{u} \cdot \nabla \mathbf{u} >$$

$$\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \nabla \mathbf{p} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{u} = -(\mathbf{u} \cdot \nabla \mathbf{u} - < \mathbf{u} \cdot \nabla \mathbf{u} >)$$

$$\nabla \cdot \mathbf{U} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = \mathbf{0}$$



Restricted Nonlinear (RNL) Equations

- RNL is a SSD model closed at second order
- RNL retains the full streamwise mean dynamics (first cumulant) and obtains the perturbation Reynolds stress from the perturbation covariance (second cumulant)
- The perturbation covariance is approximated with a finite or infinite ensemble of perturbation equations sharing the same mean flow
- (RNL_{∞} if an infinite ensemble is employed)

Restricted Nonlinear (RNL) Equations

Mean equation:

$$\mathbf{U_t} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{1}{R} \Delta \mathbf{U} = - < \mathbf{u} \cdot \nabla \mathbf{u} > \equiv \mathbf{L}(\mathbf{C})$$

When using an infinite ensemble of perturbations the covariance solves the time dependent Lyapunov equation:

$$\mathbf{C_t} = \mathbf{A}(\mathbf{U})\mathbf{C} + \mathbf{C}\mathbf{A}^{\dagger}(\mathbf{U}) + \mathbf{Q}$$

$$\mathbf{U_t + U \cdot \nabla U + \nabla P - \frac{1}{R}\Delta U = L(C)}$$

$$\mathbf{C_t = A(U)C + C(A(U))^\dagger + Q}$$

- The RNL system is an SSD for the co-evolution of the state variables (U(t), C(t)).

- RNL system is deterministic, autonomous and nonlinear.

- Trajectories of (U(t), C(t)) may converge to a fixed point, a limit cycle or a chaotic attractor.

- Bifurcations can be explored by linear perturbation analysis of the fixed points in this system.

- Chaos in RNL corresponds to chaos not of a realization of turbulence but rather to chaos of a statistical state trajectory.

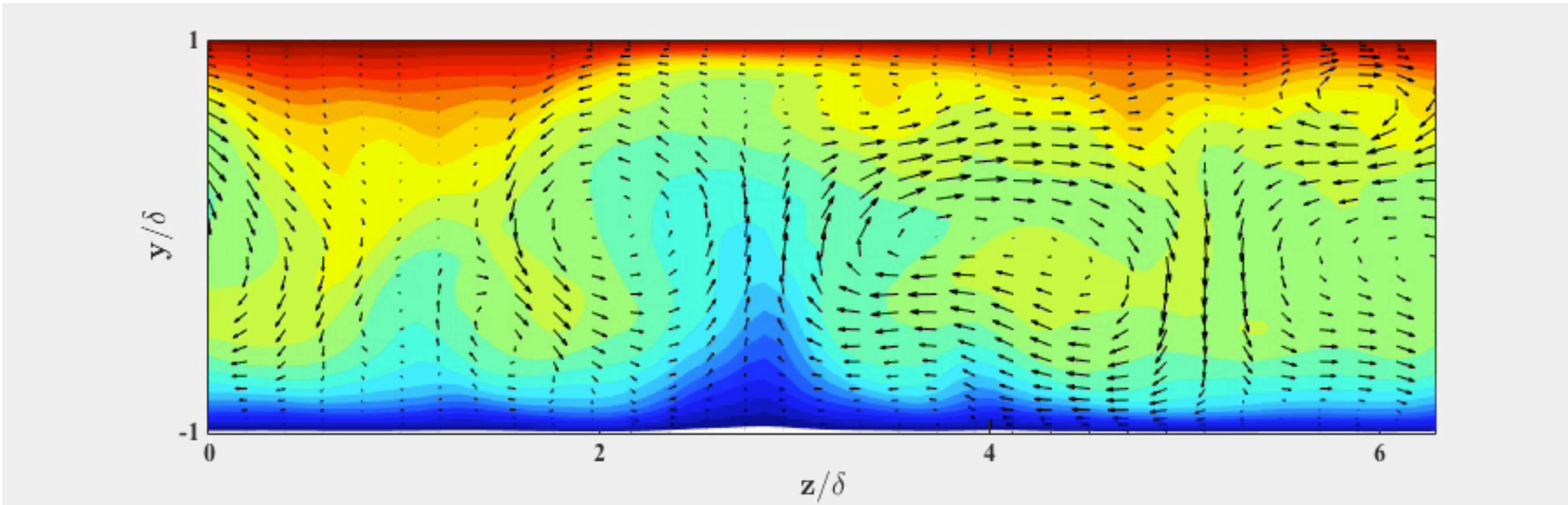
$$\mathbf{U_t} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U} = \mathbf{L}(\mathbf{C})$$

$$\mathbf{C_t} = \mathbf{A}(\mathbf{U})\mathbf{C} + \mathbf{C}(\mathbf{A}(\mathbf{U}))^{\dagger} + \mathbf{Q}$$

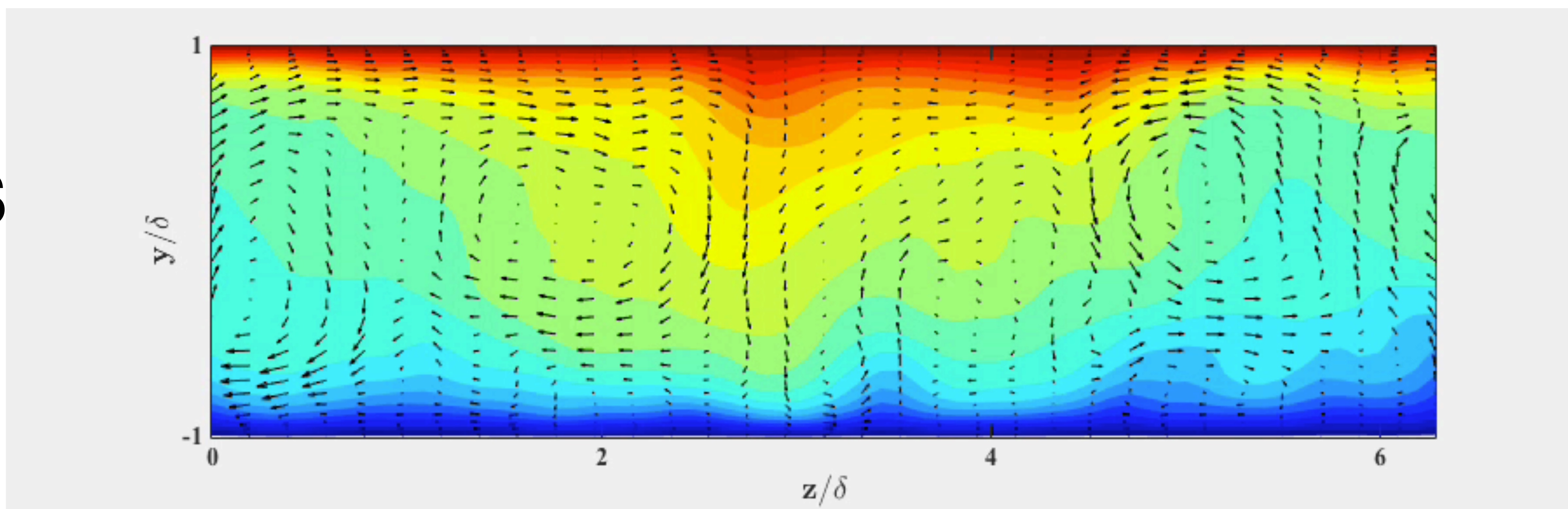
Surprisingly, it turns out that Q (the stochastic parameterization for perturbation-perturbation nonlinearity) can be set to zero in this closure and the turbulence nonetheless self-sustains.

RNL supports turbulence similar to DNS

RNL



DNS



shown is kx=0 component Lx= 4π, R=1000:

Simulations based on ‘channelflow’ code

[Peyret 2002, Gibson 2007]

- Whereas DNS turbulence is complex and not well understood in contrast RNL turbulence is completely characterized.
- RNL turbulence is simple (rank1) while DNS turbulence is of high rank.
- This reduction in complexity is spontaneous and understood.
- The spontaneous reduction in complexity of the turbulence is accompanied by a natural reduction in the number of streamwise modes supporting the turbulence.

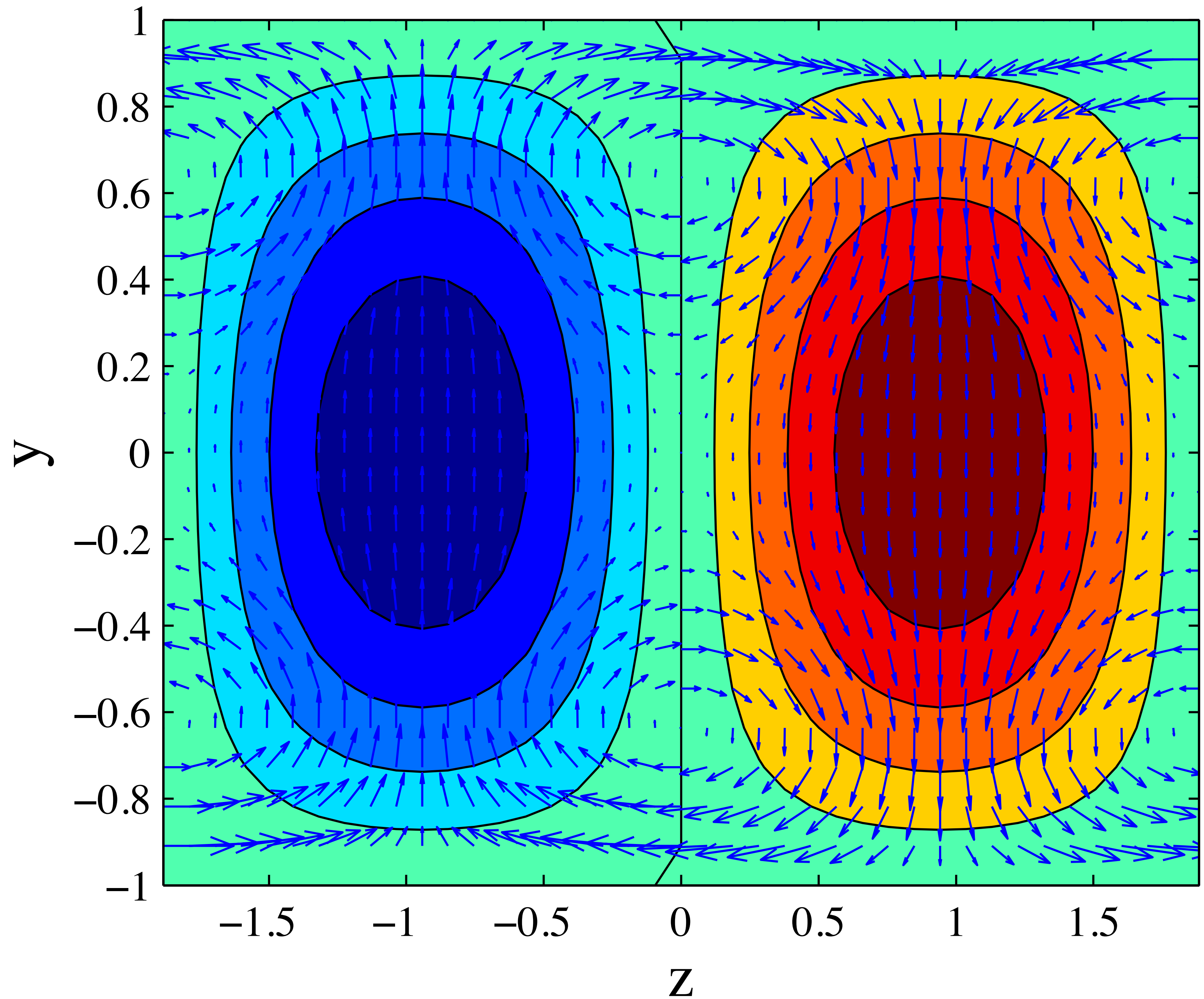
Consider RNL_∞

$$\mathbf{U_t} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U} = \mathbf{L}(\mathbf{C})$$

$$\mathbf{C_t} = \mathbf{A}(\mathbf{U})\mathbf{C} + \mathbf{C}(\mathbf{A}(\mathbf{U}))^\dagger + \mathbf{Q}$$

- By itself the second of these equations constitutes a stochastic turbulence model (STM) for the perturbations.
- We can exploit this STM to understand a fundamental mechanism of wall-turbulence dynamics.

Consider perturbing a stochastically maintained turbulence in Couette flow with a small streak and solving the STM for the forcing of the roll (V,W) that results.



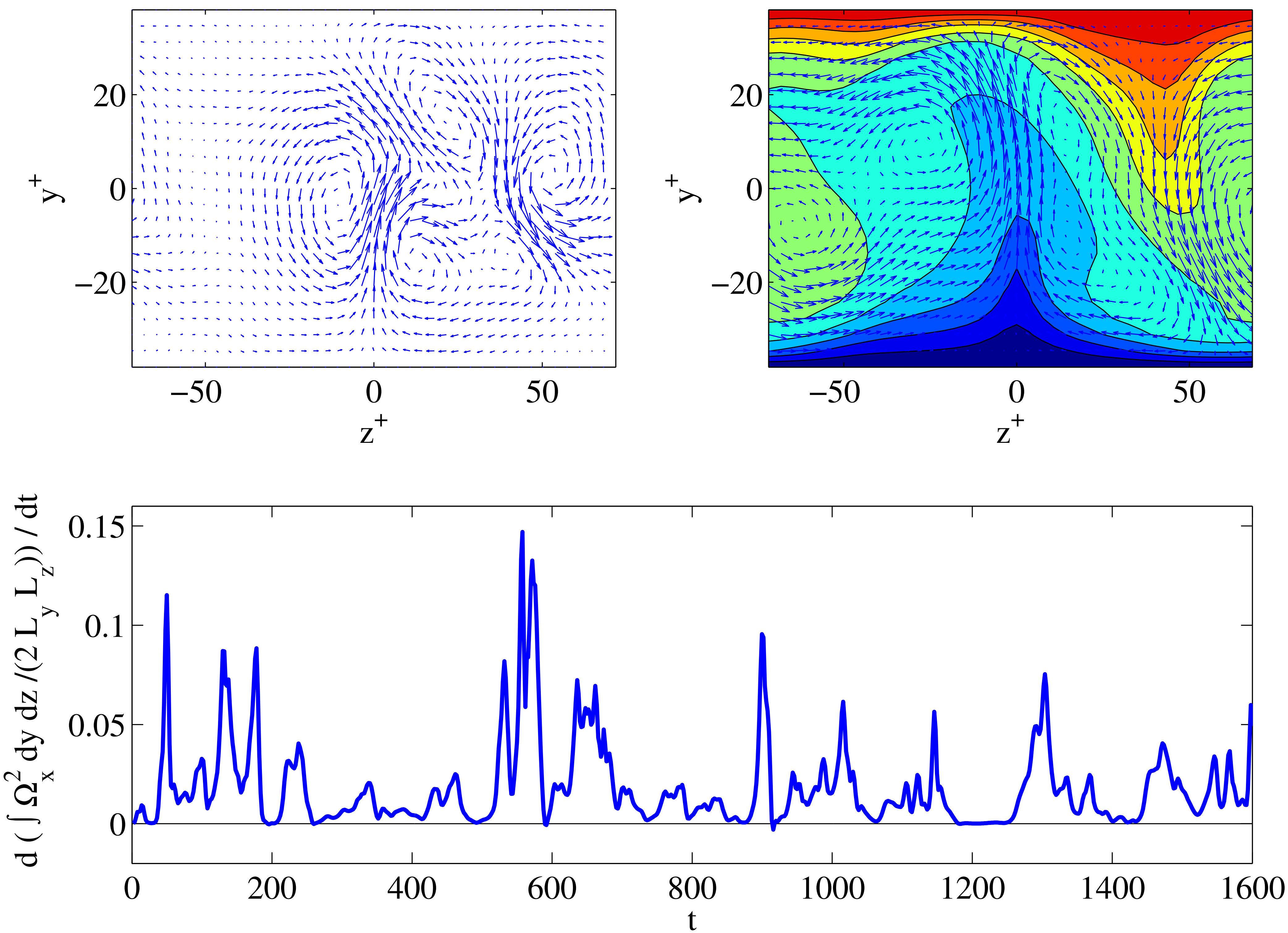
R=600

Reynolds stresses are organized by the imposed streak to produce lift up configured to amplify the imposed streak.

- Can linearize the RNL system about the Couette flow equilibrium and find the unstable eigenfunctions which are roll/streak structures that grow exponentially in free stream turbulence. These are intrinsically SSD instabilities.

- However, the interesting result for our purposes is not these eigenfunctions but rather the implied existence of a universal fast mechanism supporting the roll/streak structure.

Forcing of the streak by its organization of perturbation Reynolds stresses occurs on the advective time scale and underlies maintenance of the roll/streak structure in turbulent flows.



- It remains to understand the mechanism maintaining the perturbation variance as well as how the system is regulated to maintain a stable statistical equilibrium state
- the first results from parametric instability of the time-dependent streak
- the second results from rapid optimal perturbation growth on highly inflected streaks

Review of Parametric Instability

The undamped harmonic oscillator in energy coordinates with restoring force perturbation ω' has dynamics:

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -(1+\omega') & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

Just as in the NS equations for turbulent flow the instantaneous growth rates and directions are eigenvalues and eigenvectors of the linearized dynamics $\frac{A+A'}{2}$ which in this case are $\pm\frac{\omega'}{2}$ and $(-1,1);(1,1)$ respectively.

As the solution vector rotates it grows when aligned along (-1,1) and decays when aligned along (1,1) averaging to zero net growth.

If the restoring force perturbation is applied as the solution vector passes (-1, 1) and removed as it passes (1, 1) the solution vector does not lose passing (1,1) its gain on passing (-1,1) and will be exponentially destabilized by this time dependent restoring force despite being stable at each instant.

Conceptually this is the mechanism maintaining turbulence (and the reason turbulence is necessarily time-dependent).

- This is the familiar mechanism of the Mathieu equation by which the time dependent harmonic oscillator is destabilized.

- This mechanism requires resonant forcing and it is not the mechanism producing parametric growth in turbulent boundary layers.

- The parametric growth mechanism in wall-turbulence is that of Oseledets (1968): it is the stochastic parametric mechanism that produces the unstable Lyapunov spectrum in random matrix dynamics.

- This mechanism depends on the convexity of the exponential propagator and can be understood by considering the stretching of a material line in a turbulent nondivergent fluid.

- The compelling similarity of RNL and NS turbulence and the great simplicity of RNL dynamics motivates closer study of the mechanisms underlying RNL turbulence.

- The method we adopt is to synchronize two RNL systems so that the perturbation dynamics can be studied in isolation.

Consider an RNL turbulence self-sustaining without stochastic forcing (Q=0):

$$\partial_t \mathbf{U_a} + \mathbf{U_a} \cdot \nabla \mathbf{U_a} + \nabla \mathbf{P_a} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U_a} = \mathbf{L}(\mathbf{C_a})$$

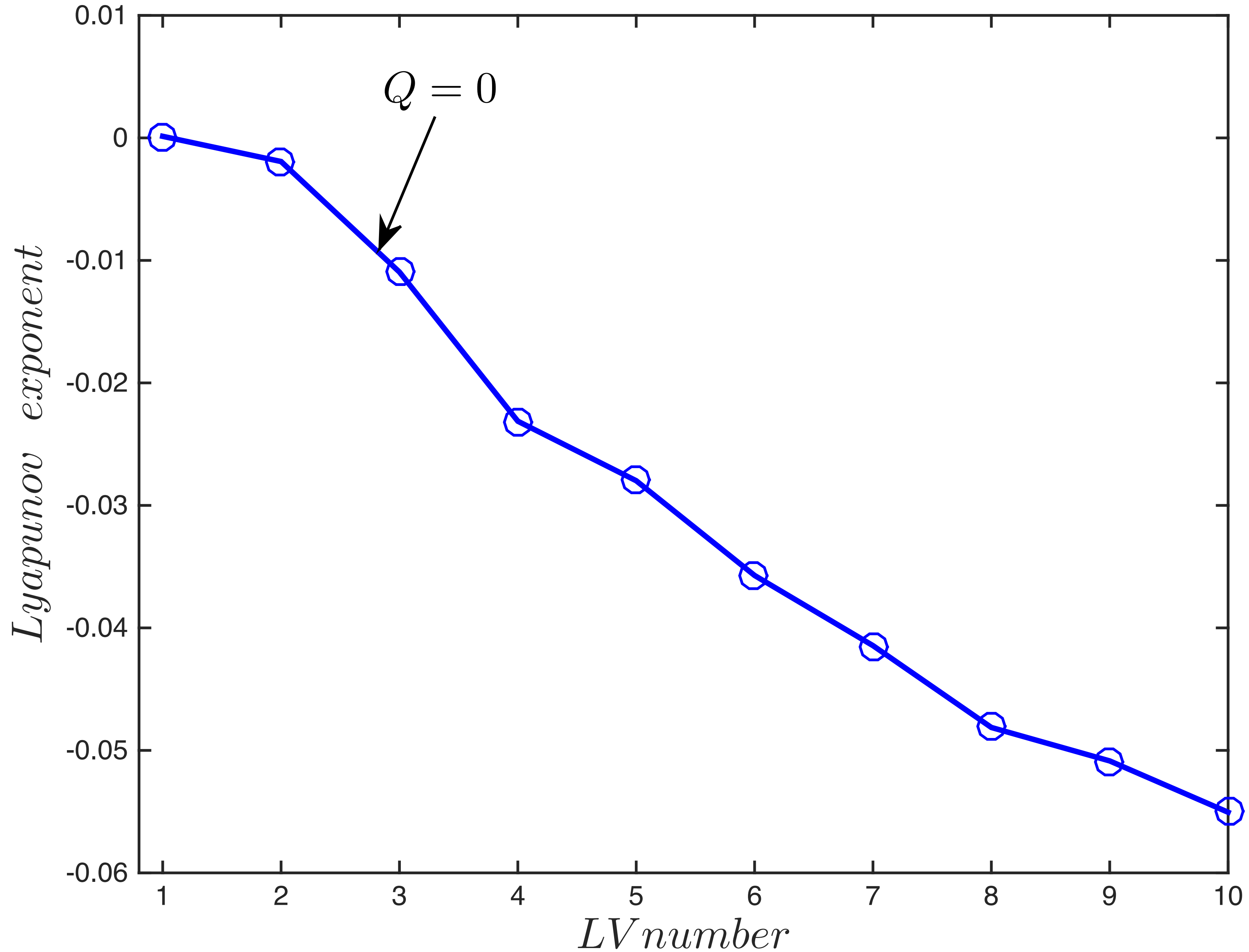
$$\partial_t \mathbf{C_a} = \mathbf{A}(\mathbf{U_a})\mathbf{C_a} + \mathbf{C_a}(\mathbf{A}(\mathbf{U_a}))^\dagger$$

Now impose the streak alone from this turbulence on the perturbations dynamics of a second RNL system initialized with a random full rank covariance:

$$\partial_t \mathbf{C_b} = \mathbf{A}(\mathbf{U_a})\mathbf{C_b} + \mathbf{C_b}(\mathbf{A}(\mathbf{U_a}))^\dagger$$

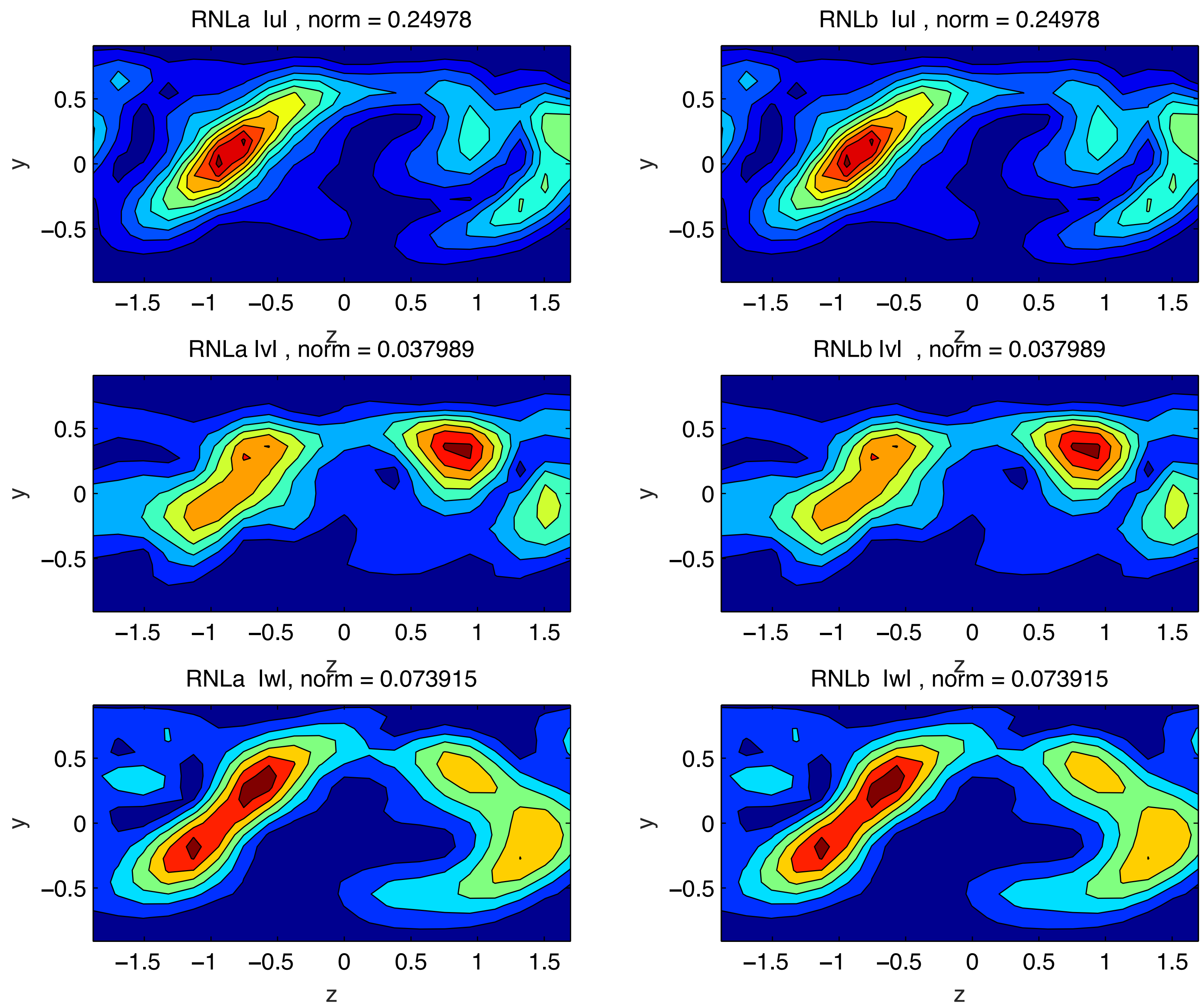
- Examination of the perturbation dynamics reveals a linear essentially stochastic time-dependent system so the asymptotic structure of the perturbation field is the first Lyapunov vector.
- Given that the Lyapunov vector is a component of the state trajectory the associated Lyapunov exponent is necessarily zero.

Synchronized system perturbation field converges to the first Lyapunov vector of the primary system.



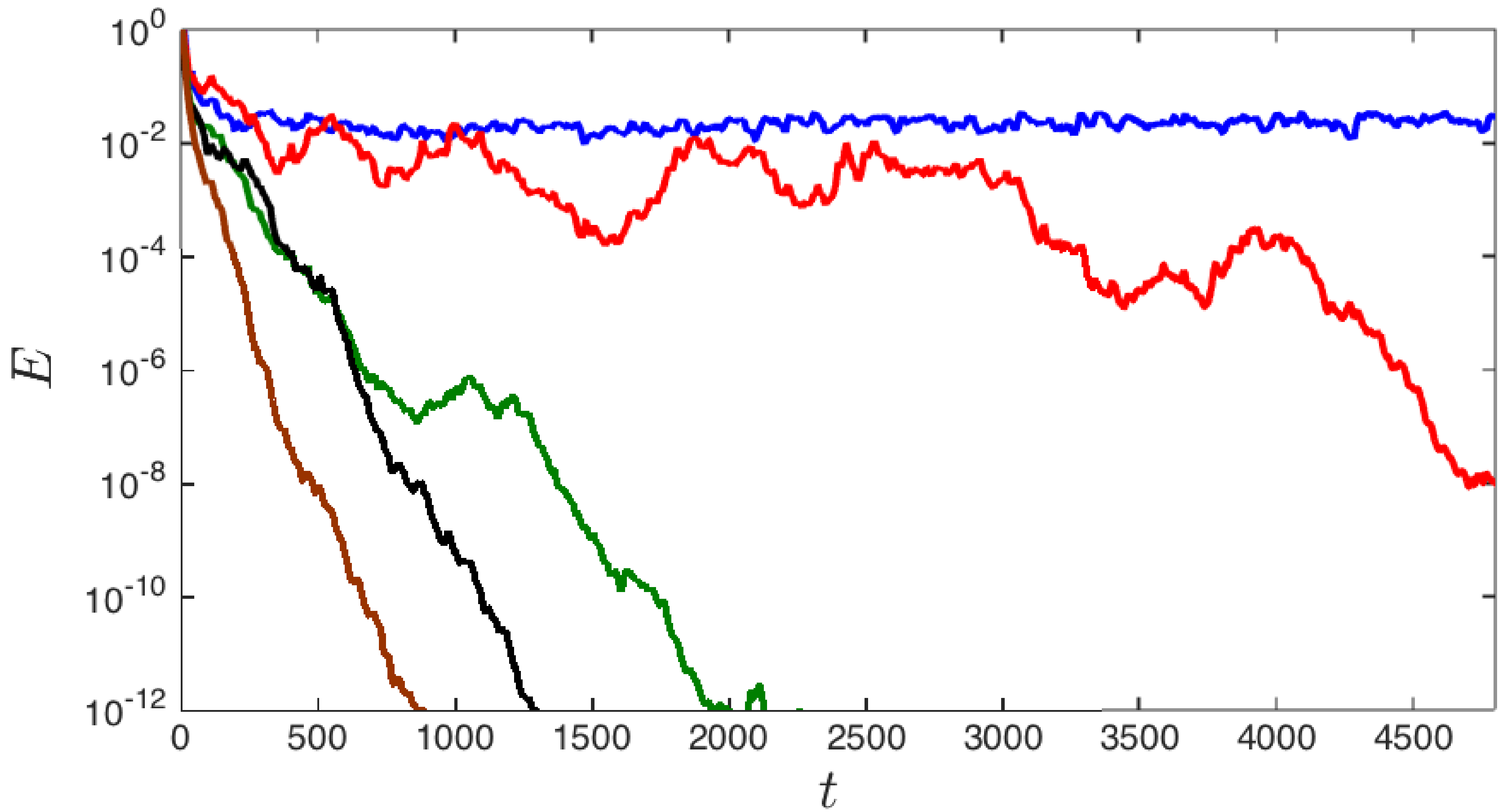
The channel has length Lx = 1.75, width Lz = 1.2 and R = 600; the single streamwise wavenumber k=2 pi/Lx is retained by the dynamics of LV1

Synchronized system perturbation field converges to the first Lyapunov vector of the primary system which supports only a single streamwise mode greatly reducing the complexity.



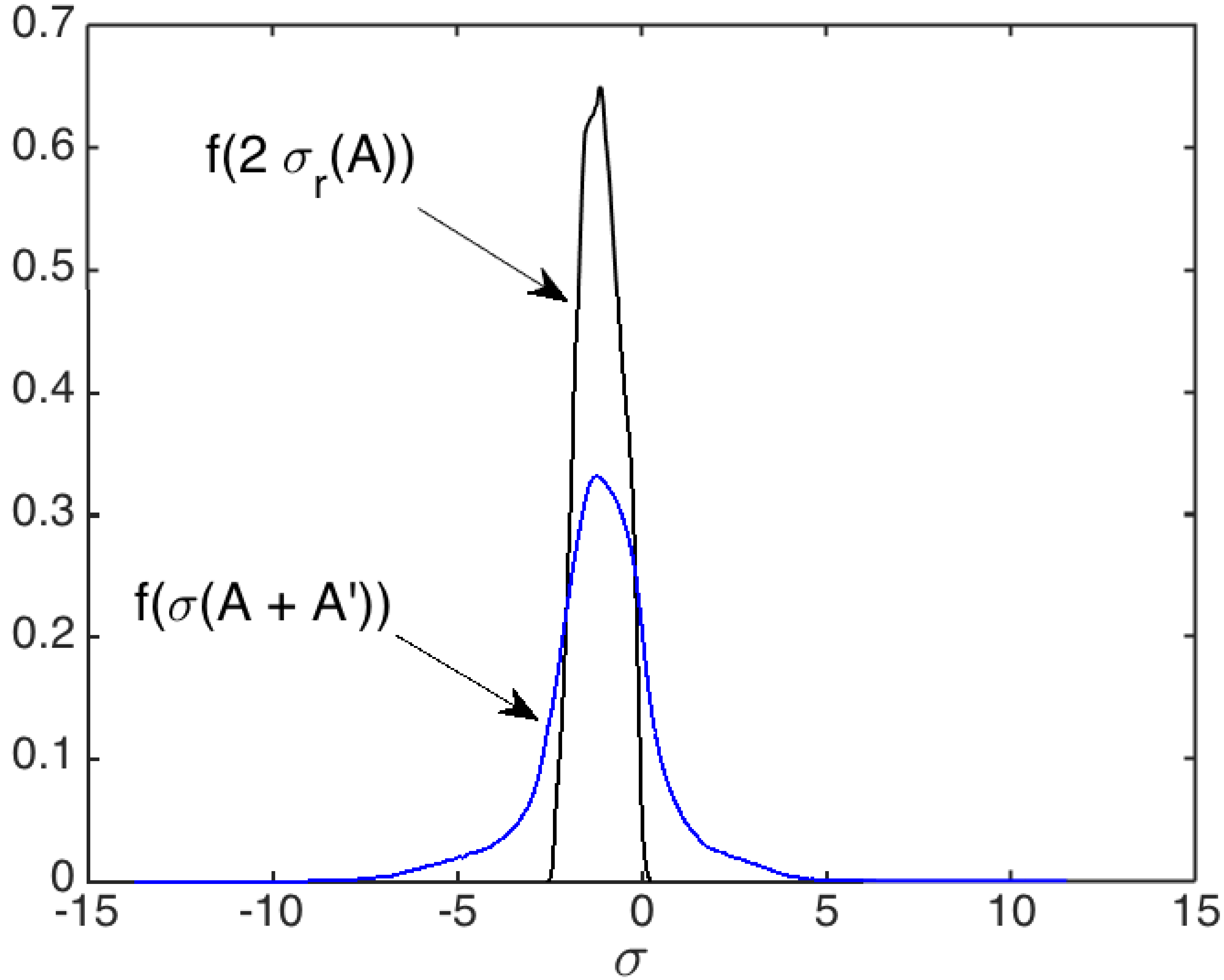
The channel has length Lx = 1.75, width Lz = 1.2 and R = 600; the single streamwise wavenumber k=2 pi/Lx is retained by the dynamics.

Stochastic forcing of the synchronized dynamics reveals the existence of both the trajectory Lyapunov vector (active subspace) as well as the other Lyapunov vectors with negative exponents (passive subspace).

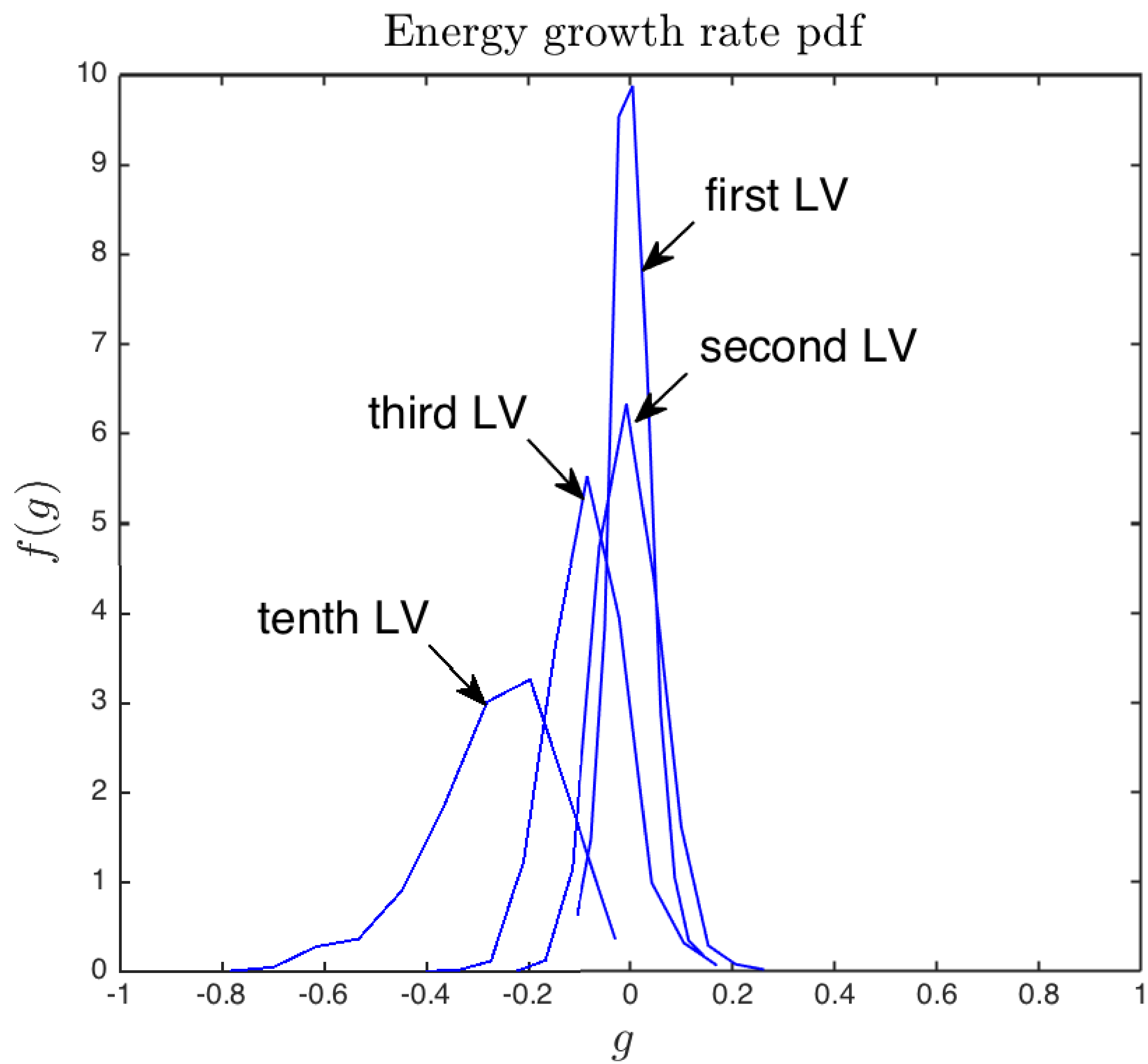


- We wish to examine the mechanism by which LV1 is maintained and regulated.
- Method is to diagnose the synchronized system dynamics.

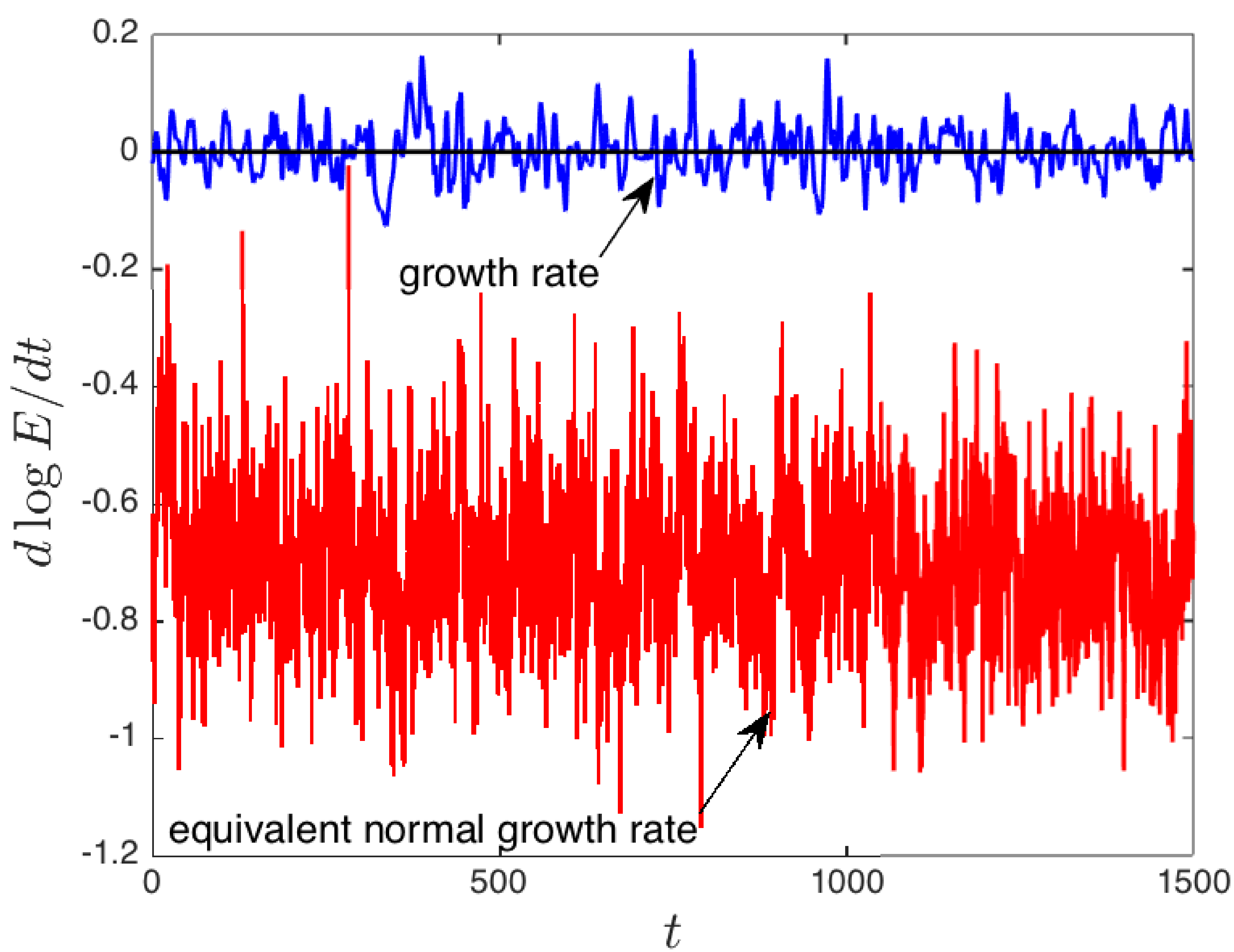
- Instantaneous growth rate possible for a perturbation (t=[0,5000]).
- Note instability boundary.



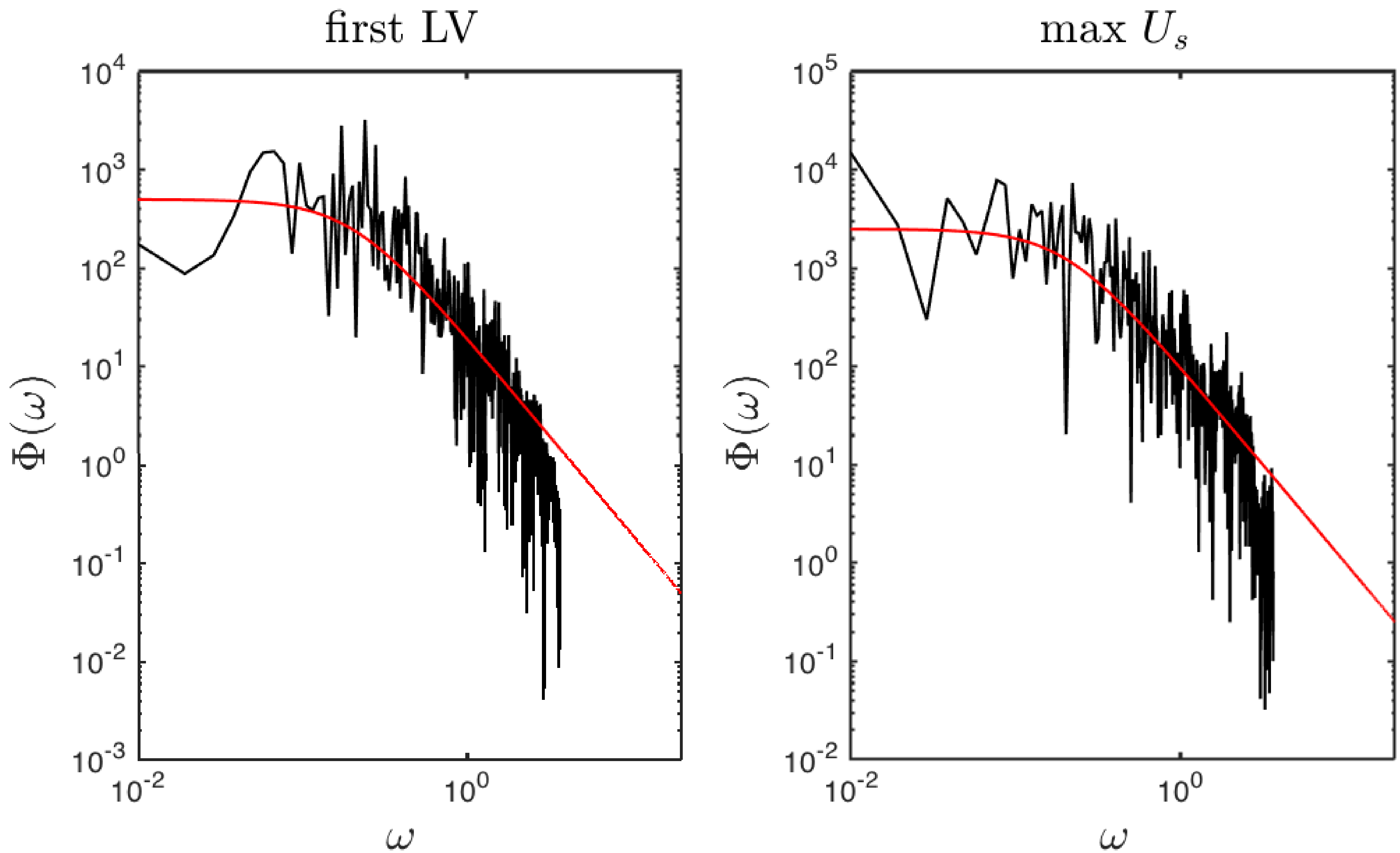
Instantaneous growth rate achieved by Lyapunov state vectors (t=[0,5000]).



Partition of the instantaneous growth rate of the LV1 into modal and non-modal sources.

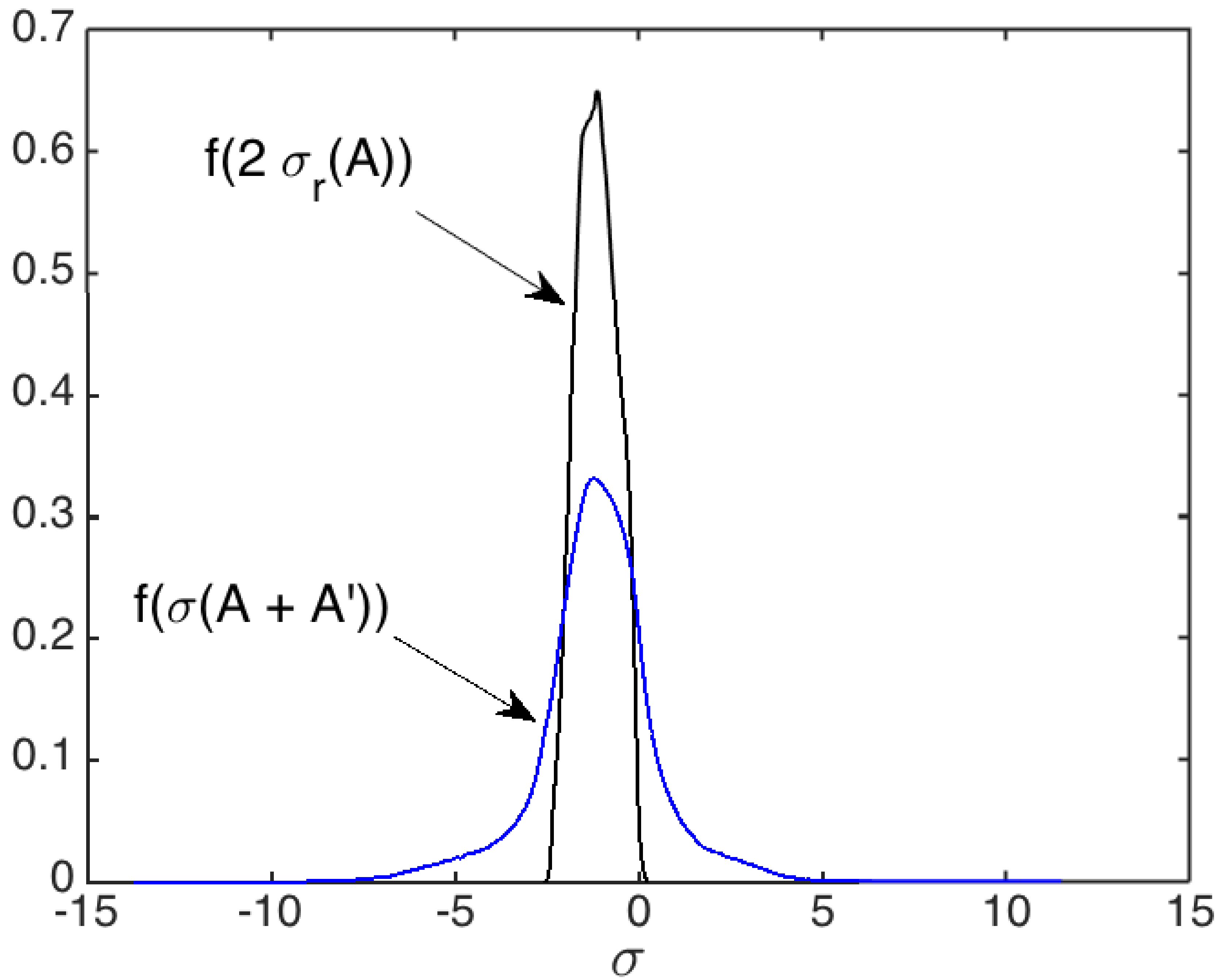


Spectra of the instantaneous growth rate of the perturbation state vector (LV1) and the mean streak reveal identical red noise processes.

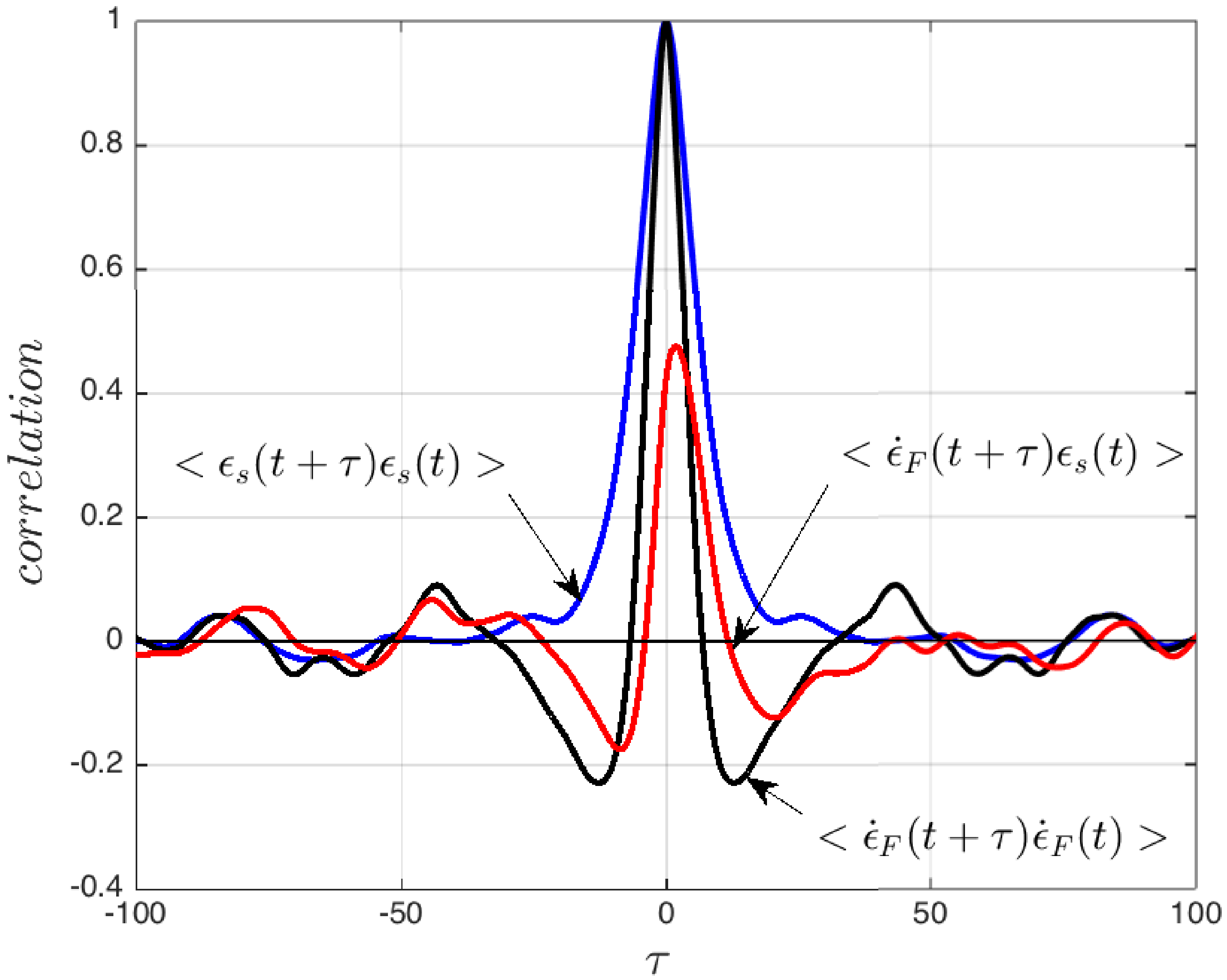


Turn now to the question of how the system is regulated to maintain a statistical steady turbulent state.

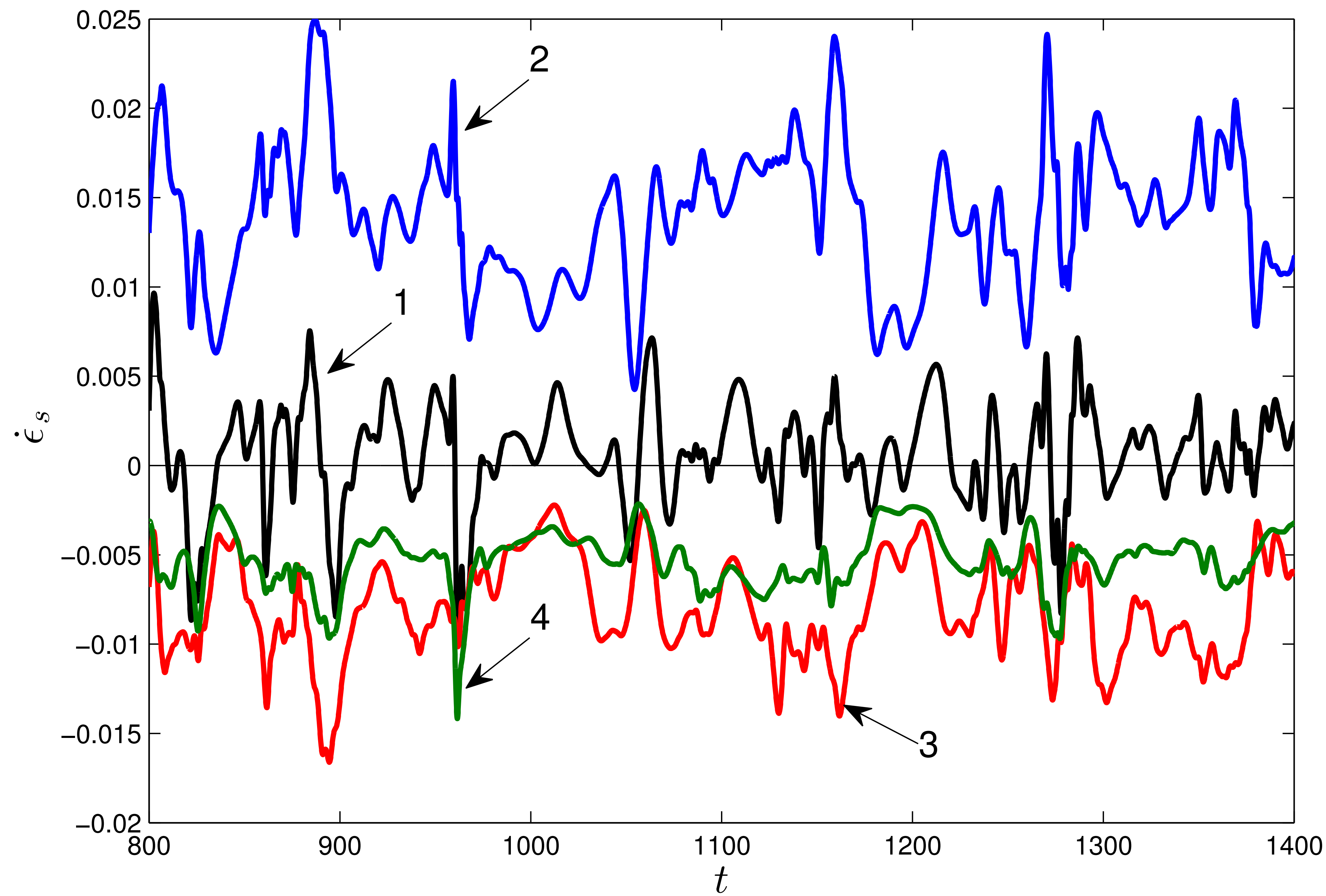
- Instantaneous growth rate possible for a perturbation (t=[5000]).
- Note instability boundary.



Auto and cross-correlation of streak amplitude and Reynolds stress damping reveals a time scale far shorter than that of the instability -> regulation of the streak occurs on the advective time scale.



The statistical state is regulated by a rapid perturbation Reynolds stress mediated feedback associated with streak inflection which is in balance with roll-induced lift-up. The mechanism of this regulation is adjoint mode growth which occurs on the advective time scale.



streak growth rate (1), lift-up (2), Reynolds stress (3), damping (4)

Conclusions

- SSD provides a powerful tool for studying the dynamics of turbulence.
- RNL model is a second order SSD model that maintains highly realistic turbulence.
- The dynamics of the RNL system are directly connected to NS dynamics.
- The RNL dynamics are naturally minimal.
- The RNL dynamics are completely characterized analytically.
- RNL turbulence is maintained by the stochastic parametric growth mechanism which is a universal property of time-dependent dynamical systems.
- RNL turbulence is regulated by adjoint mode growth on the advective time scale.