Lie-group derivation of a bulk flow scaling for wall bounded turbulence

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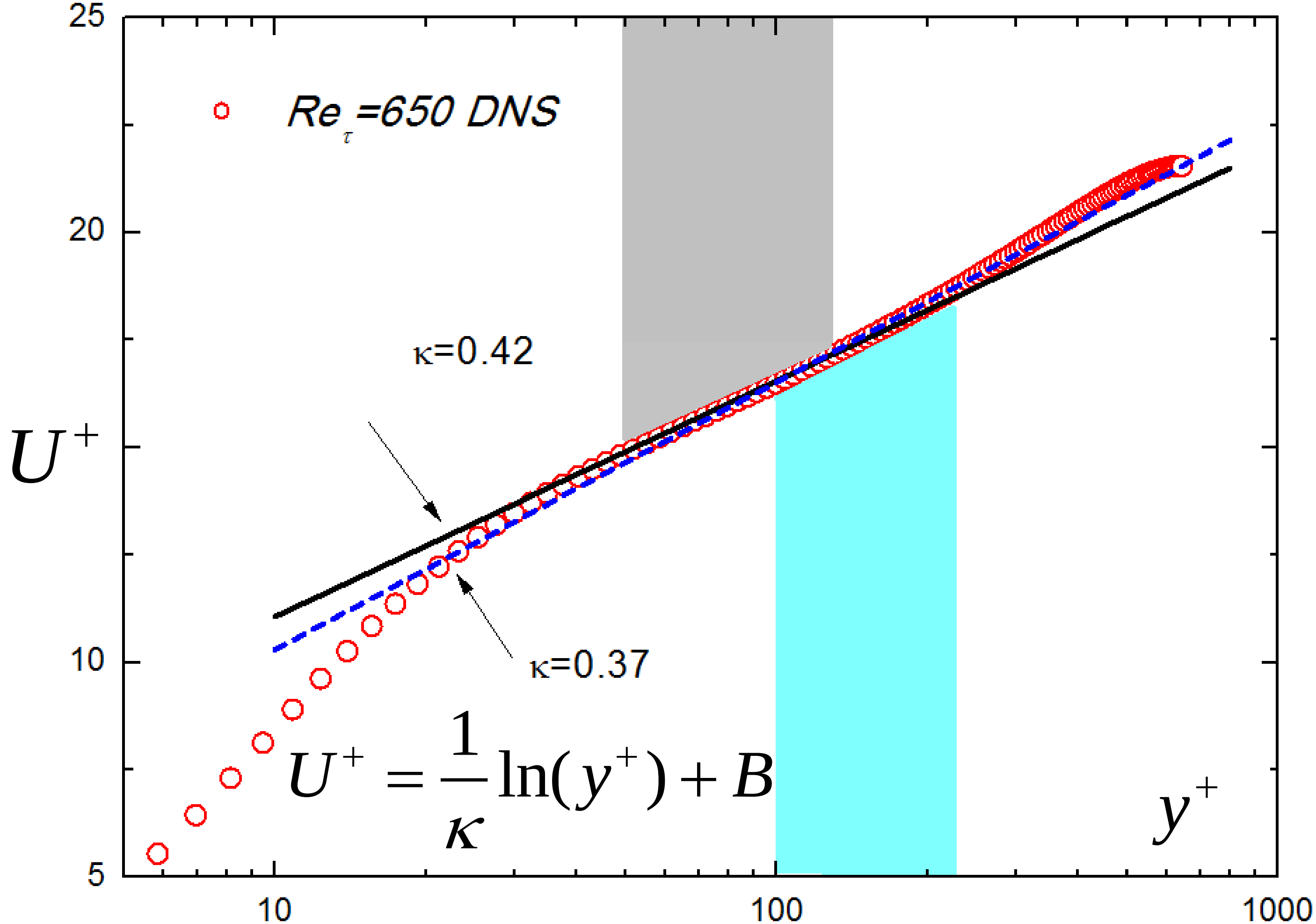
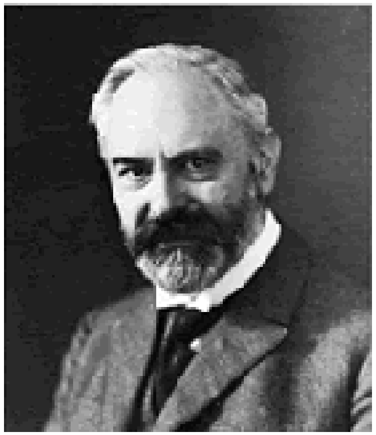
+also, **Texas Tech University**

Thanks for data sharing

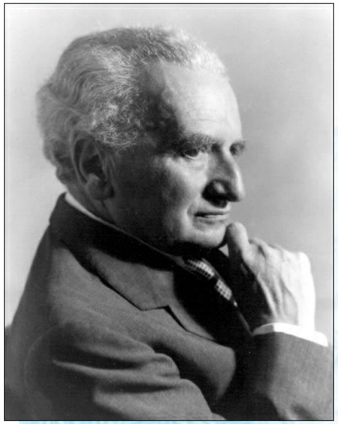
- **Direct numerical simulation (DNS) data**
  - Channel (CH) – Jimenez etc.;
    – Iwamoto etc.;
  - Pipe - Wu & Moin;
  - Turbulent boundary layer (TBL) - KTH
- **Experimental (EXP) data**
  - Princeton smooth and rough Pipe
  - Melbourne Channel & TBL
  - Lille TBL (Stanislas etc.;)
  - Stanford TBL (Degraff & Eaton)

Log-law for the mean velocity profile (MVP)





Ludwig Prandtl
1875 –1953



Theodore von Kármán
1881-1963

Different fitting domain leads to different κ

Channel DNS data from Iwamoto et., 2006

Challenge on log-law and Karman constant

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Science 17 May 1996:
Vol. 272 no. 5264 p. 951
DOI: 10.1126/science.272.5264.951

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RESEARCH NEWS

MATHEMATICS

A New Theory of Turbulence Causes a Stir Among Experts

Barry Cipra

Power law

1993, Barenblatt

1996, Science News

In 2006, 36th AIAA meeting, P. Spalart：

- **We lost the Karman constant!**
- **If Karman constant is different in TBL and Pipe, *I will quit*!**

How to resolve the debates? Two perspective:

- **A complete description of MVP (MCN, 2007-2008, PoF)**
- **Beyond the mean momentum level (Marusic etc., 2012, JFM)**

Smits, Mckeon & Marusic, *Annu. Rev. Fluid Mech. (2011)*

Budget of mean kinetic energy equation (MKE)

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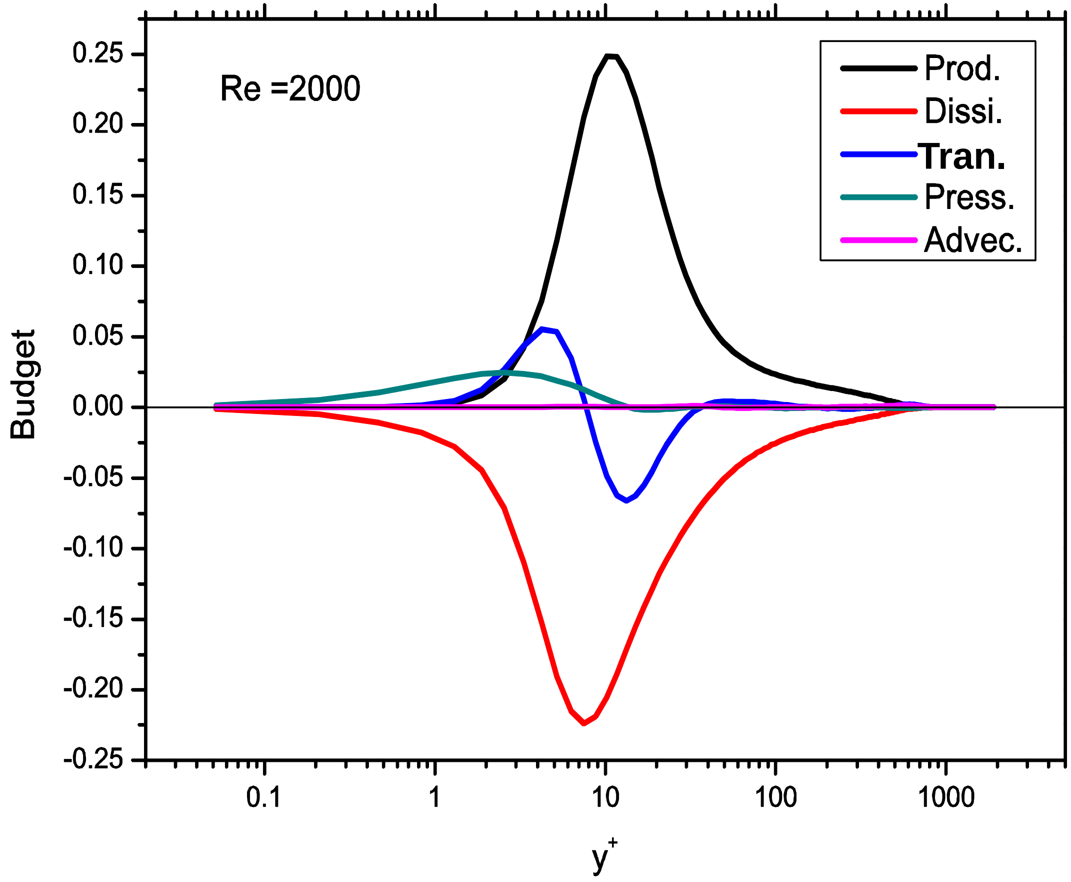
MKE: $$S\,W + \Pi_{press.} + T_{trans.} + A_{advec.} = \varepsilon$$



$$S = \nu dU / dy$$

Mean shear stress

$$W = -\langle uv \rangle$$

Reynolds stress

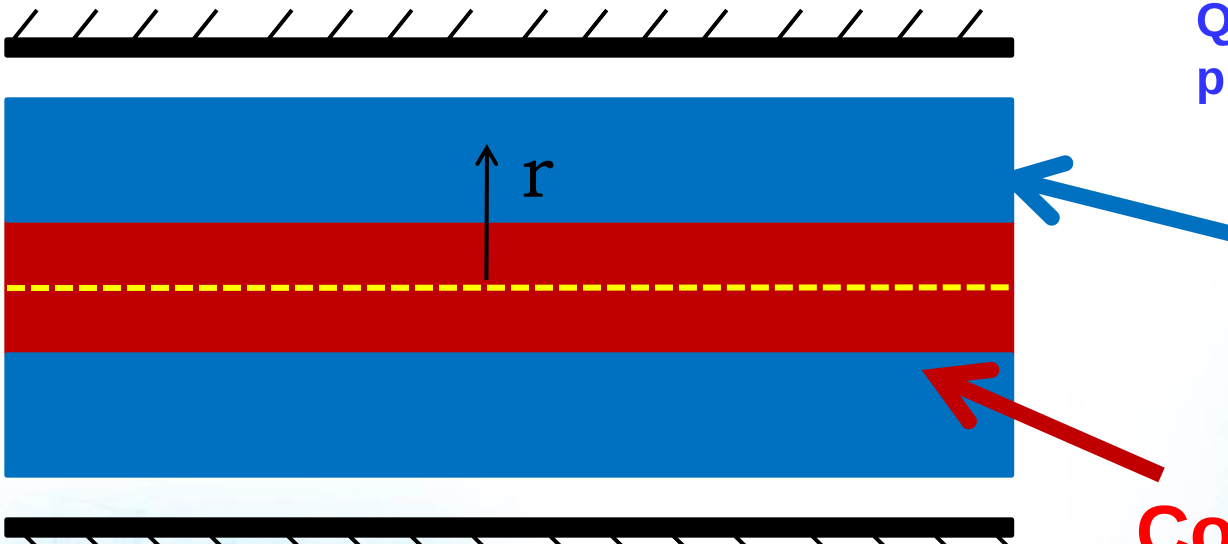
$$\varepsilon = \nu\,|\nabla u|^2 + \nu d^2 k / dy^2$$

Turbulent dissipation

Channel: Bulk & Core layer

$$S\,W + \Pi_{press.} + T_{trans.} + A_{advec.} = \varepsilon$$

Wall



Wall

Quasi Balance between production and dissipation

$$SW \approx \varepsilon$$

Bulk flow

Core layer

$$T_{trans.} \approx \varepsilon$$

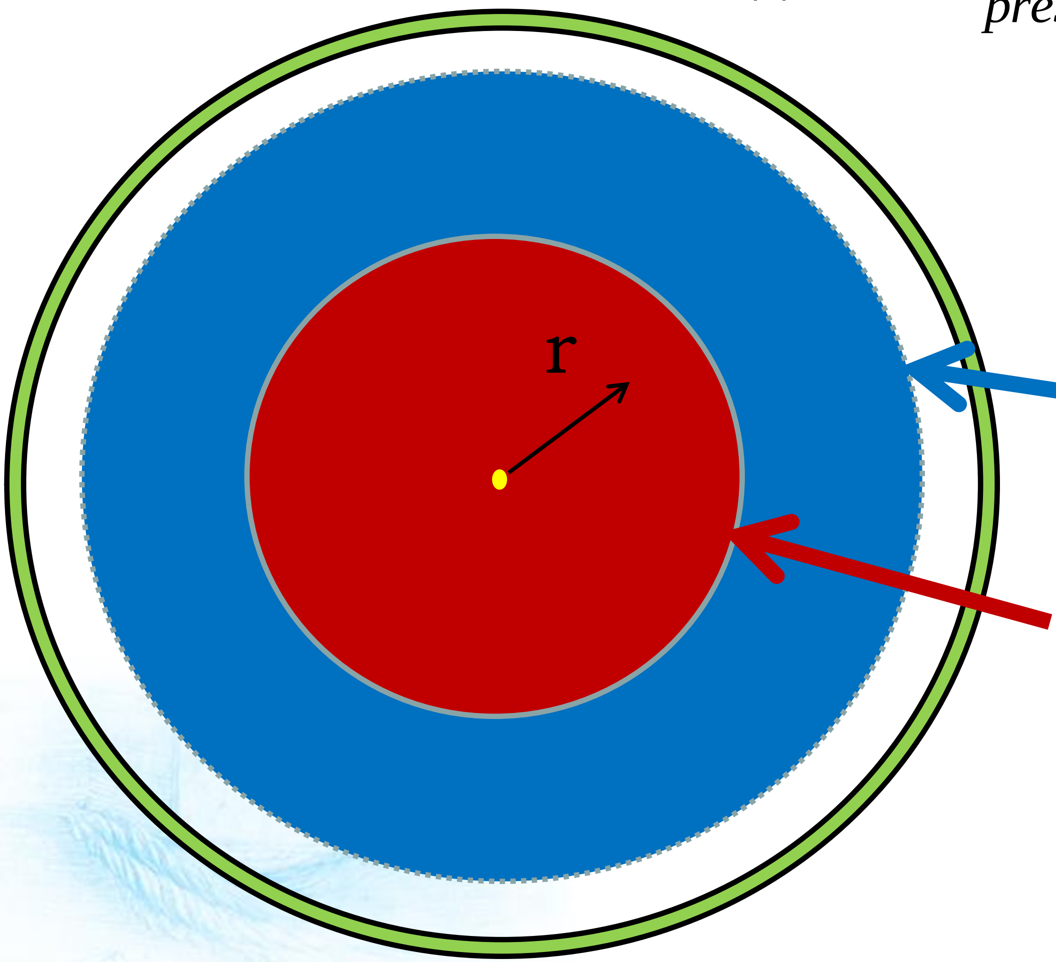
half height δ

wall distance $y = (1-r)\delta$

Pipe: Bulk & Core layer

$$S\,W + \Pi_{press.} + T_{trans.} + A_{advec.} = \varepsilon$$

Wall



Wall

Quasi Balance between production and dissipation

$$SW \approx \varepsilon$$

Bulk flow

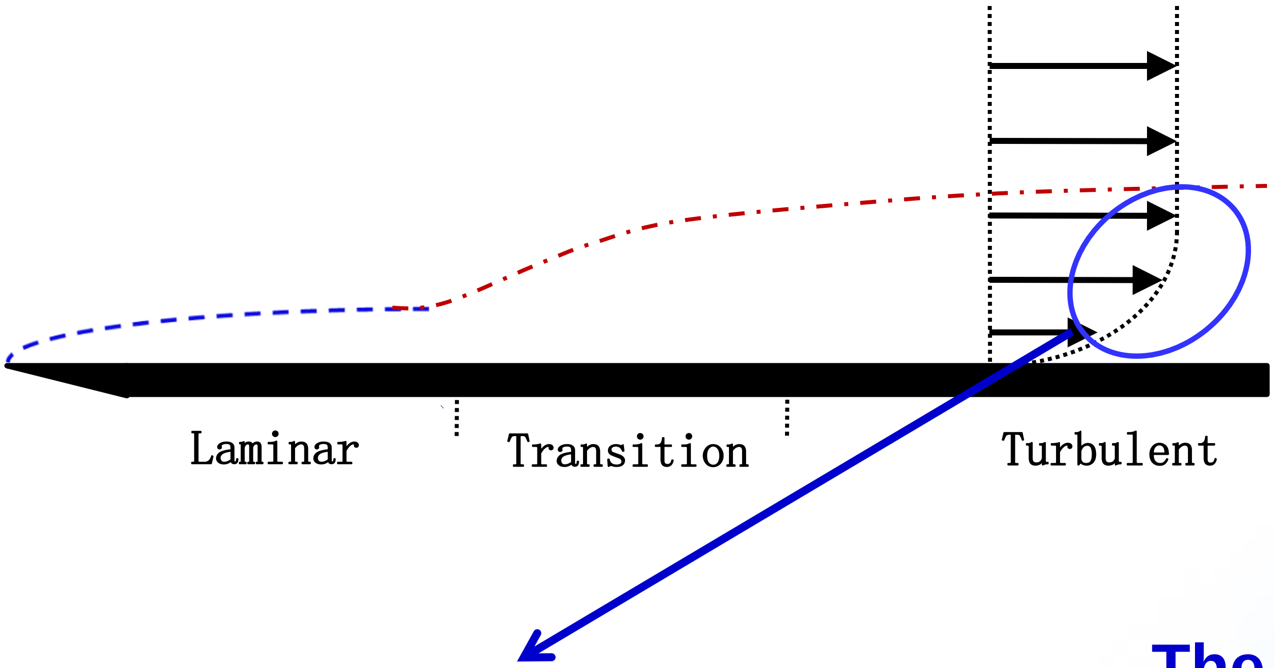
Core layer

$$T_{trans.} \approx \varepsilon$$

wall distance $y = (1 - r)\delta$

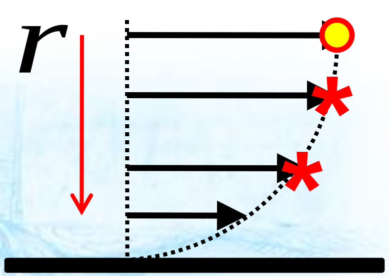
Pipe radius δ

TBL: Bulk



Bulk of TBL

$$SW \approx \varepsilon$$



$$\delta = \alpha \delta_{99} \quad (\alpha \to 1)$$

A location similar to centreline of a channel or pipe.

The quasi-balance region extends (almost) to boundary layer edge (δ_{99}), and no centre core.

wall distance $y = (1 - r)\delta$

Length function quantifying bulk and core

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- **Mixing length function:**

$$\ell_M = \frac{\sqrt{-\langle uv \rangle}}{dU/dy}$$

$$\ell_M^{Pipe}/\delta = \frac{\kappa}{5}(1-r^5)\Theta^{1/4}(r)$$

$$\ell_M^{CH}/\delta = \frac{\kappa}{4}(1-r^4)\Theta^{1/4}(r)$$

$$\ell_M^{TBL}/\delta = \frac{\kappa}{4}(1-r^4)$$

Extending the linear scaling (Prandtl, 1925) into entire outer flow:

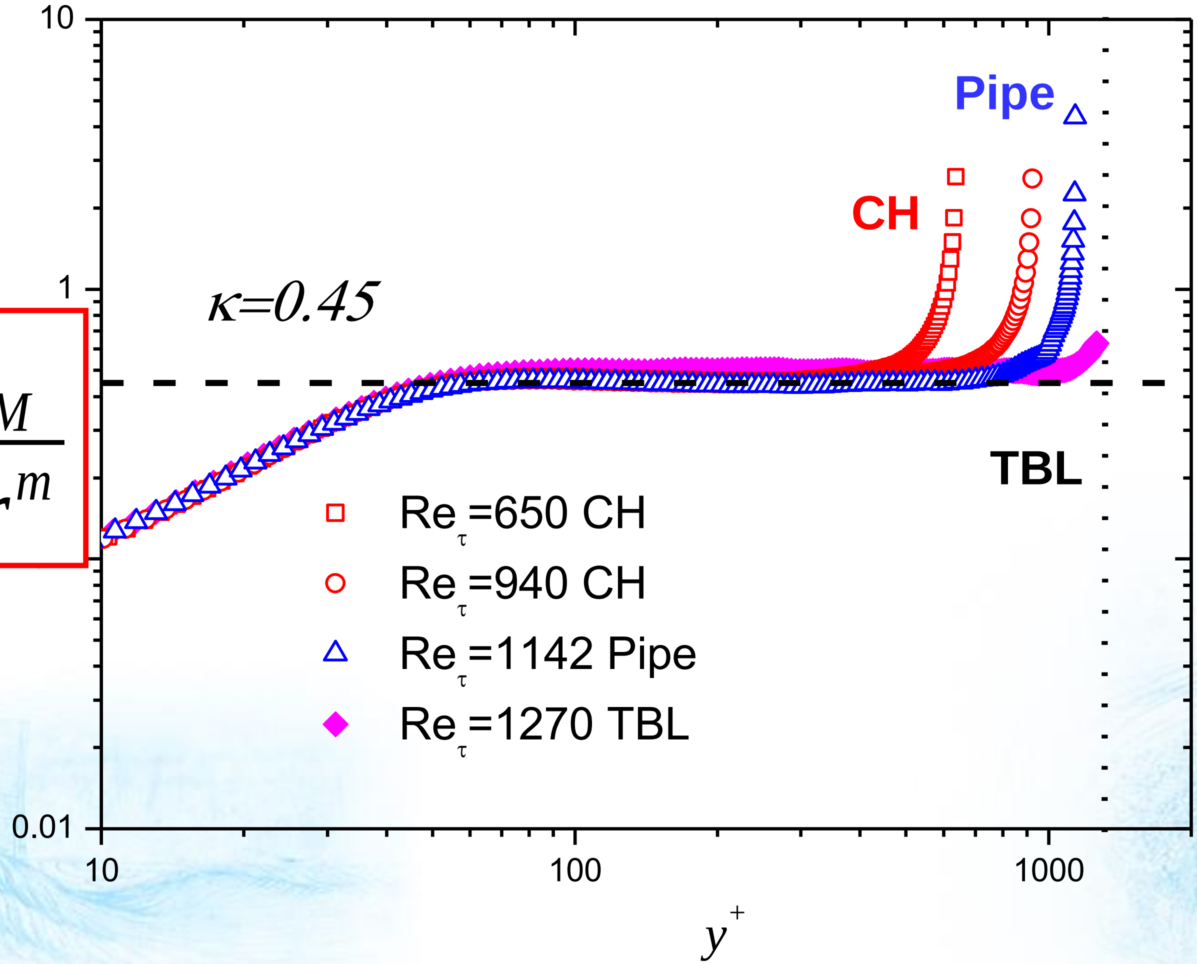
$$\ell_M \rightarrow \kappa(1-r)\delta = \kappa y \quad \textbf{as} \quad \mathrm{r} \rightarrow 1$$

Ratio of turbulent dissipation and production

$$\Theta(r) \equiv (\varepsilon / SW) = [1+(r_c/r)^2]/(1+r_c^2)$$

Test by DNS data

$$\frac{m\ell_M}{1-r^m}$$



In bulk flow

$$\ell_M \approx \kappa\left(1-r^4\right)/4$$

$$\ell_M \approx \kappa\left(1-r^5\right)/5$$

In core layer

$$\ell_M \propto r^{-1/2}$$

Question: can we obtain these scaling laws from NS (RANS) equations?

- **Symmetry transformation for NS equations**

$$\frac{\partial u_i}{\partial x_i}=0 \qquad \Rightarrow \qquad \frac{\partial u_i^*}{\partial x_i^*}=0$$

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=\nu\frac{\partial^2 u_i}{\partial x_k^2}-\frac{\partial p}{\partial x_i} \qquad \frac{\partial u_i^*}{\partial t^*}+u_k^*\frac{\partial u_i^*}{\partial x_k^*}=\nu\frac{\partial^2 u_i^*}{\partial x_k^{*2}}-\frac{\partial p^*}{\partial x_i^*}$$

In the text books of Frisch (1995), Pope (2000) and Cantwell (2002)

$$t,\vec{r},\vec{u}\rightarrow t,\vec{r}+\vec{\rho},\vec{u}, \quad \vec{\rho}\in R^3 \qquad t,\vec{r},\vec{u}\rightarrow t,-\vec{r},-\vec{u}.$$

$$t,\vec{r},\vec{u}\rightarrow t+\tau,\vec{r},\vec{u}, \quad \tau\in R \qquad t,\vec{r},\vec{u}\rightarrow t,A\vec{r},A\vec{u}, \quad A\in SO(R^3)$$

$$t,\vec{r},\vec{u}\rightarrow t,\vec{r}+\vec{V}t,\vec{u}+\vec{V}, \quad \vec{V}\in R^3 \qquad t,\vec{r},\vec{u},\nu\rightarrow e^s t,e^{\lambda s}\vec{r},e^{(\lambda-1)s}\vec{u},e^{(2\lambda-1)s}\nu \quad s\in R,\lambda\in R.$$

Translations, Reflections, Galilean transformation, Rotations, Dilations,

Symmetry analysis with length function (parallel flow)

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Continuity equation

$$\frac{\partial \overline{u_k}}{\partial x_k} = 0$$

Reynolds stress

$$W_{ik} = \overline{u_i' u_k'}$$

Mean momentum equation

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_k}\frac{\partial \overline{u_i}}{\partial x_k} + \frac{\partial W_{ik}}{\partial x_k} = \nu \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_k} - \frac{\partial \overline{p}}{\partial x_i}$$

Introducing length functions

$$\ell_{ik} = \frac{\left|\overline{u_i' u_k'}\right|^{1/2}}{\partial_y \overline{u}} \qquad \frac{\partial \overline{u_i}}{\partial t} + \frac{\partial}{\partial x_k}\left(sign[\ell_{ik}^2 (\partial_y \overline{u})^2] \right) = \nu \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_k} - \frac{\partial \overline{p}}{\partial x_i}$$

Different dilations on coordinate x_i and on length function:

$$t, x_i, \overline{u}_i, \ell_{ik}, \nu \rightarrow e^{(3-\beta-2\alpha)\epsilon} t, e^{\epsilon} x_i, e^{\beta\epsilon} \overline{u}_i, e^{\alpha\epsilon} \ell_{ik}, e^{(2\alpha+\beta-1)\epsilon} \nu$$

Symmetry analysis with length function (TBL)

Continuity equation

$$\frac{\partial \overline{u_k}}{\partial x_k} = 0$$

Reynolds stress

$$W = -\overline{u'v'}$$

Streamwsie mean momentum equation

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{\partial W}{\partial y} + \nu \frac{\partial^2 \overline{u_i}}{\partial y^2} - \frac{\partial \overline{p}}{\partial x}$$

Introducing length functions

$$\ell_M = \frac{\left|\overline{u'v'}\right|^{1/2}}{\partial_y \overline{u}} \qquad \frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{\partial}{\partial y}\left(\ell_M \frac{\partial \overline{u}}{\partial y}\right)^2 + \nu \frac{\partial^2 \overline{u_i}}{\partial y^2} - \frac{\partial \overline{p}}{\partial x}$$

Different dilations on coordinate x_i and on length function:

$$t^* = e^{(3-\beta-2\alpha)\varepsilon} t, \quad x = e^{(3-2\alpha)\epsilon} x, \quad y = e^{\epsilon} y, \quad \overline{u}^* = e^{\beta\epsilon}\overline{u},$$

$$\overline{v}^* = e^{(\beta+2\alpha-2)\varepsilon}\overline{v}, \quad \nu^* = e^{(\beta+2\alpha-1)\epsilon}, \quad \ell_M^* = e^{\alpha\epsilon}\ell_M, \quad \overline{p}^* = e^{2\beta\epsilon}\overline{p}$$

Candidate dilation invariant solution

;

Dilation invariant $$\mathbf{I}_1 = l^* / y^{*\alpha} = l / y^\alpha$$

Dilation invariant for derivative $$\mathbf{I}_2 = \left(\frac{dl^*}{dy^*}\right) / y^{*(\alpha-1)} = \left(\frac{dl}{dy}\right) / y^{(\alpha-1)}$$

Invariant solution $$\mathbf{G}(\mathbf{I}_1, \mathbf{I}_2, \ldots) = 0$$

Case 1

$$\mathbf{I}_1 = \text{const.} \quad \Rightarrow \quad l = \mathbf{I}_1 y^\alpha \qquad \mathbf{I}_2 = \alpha \mathbf{I}_1 = \text{const.}$$

Case 2

$$\mathbf{I}_2 = \text{const.} \quad \Rightarrow \quad \frac{dl}{dy} = \mathbf{I}_2 y^{(\alpha-1)} \qquad l = (\mathbf{I}_2 / \alpha) y^\alpha + c$$

Case 3

$$\mathbf{I}_2 = \alpha \mathbf{I}_1 + c(\mathbf{I}_1)^n \quad \Rightarrow \quad l = a y^\alpha \left(1 + (y / y_c)^p\right)^{\gamma/p}$$

α **to be determined by physical consideration.**

Scaling in outer units :

Two dilation invariant for mixing length function:

$$\mathbf{I}_1 = \ell_M / r^{\alpha} \qquad \mathbf{I}_2 = \dot{\ell}_M / r^{\alpha-1}$$

Constant-dilation-invariant assumption:

Power-law:	$\mathbf{I}_1 = const.$ $\mathbf{I}_2 = const.$	$\Rightarrow \ell_M = \mathbf{I}_1 r^{\alpha}$	Core layer
Defect power-law:	$\mathbf{I}_1 \neq const.$ $\mathbf{I}_2 = const.$	$\Rightarrow \ell_M = \ell_0 + (\mathbf{I}_2 / \alpha) r^{\alpha} = \ell_0 (1 - r^{\alpha})$	Bulk flow

Prediction for mean velocity

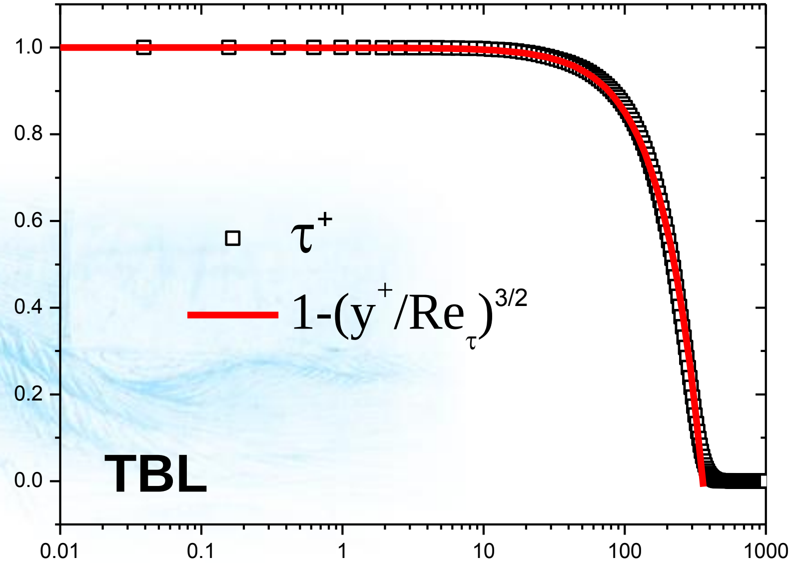
$$S^+ = \frac{dU^+}{dy^+} = \sqrt{W^+} / \ell_M^+ \approx \sqrt{\tau^+} / \ell_M^+$$

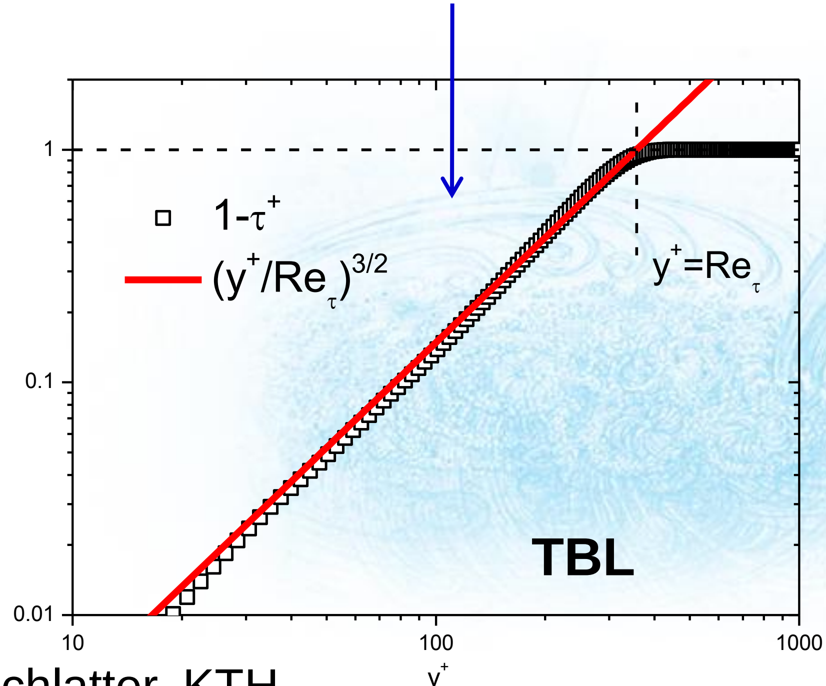
A general formula for total stress:

$$\boxed{\tau^+ = 1 - (y^+ / \mathrm{Re}_\tau)^\gamma}$$

CH & Pipe: $\gamma = 1$

TBL: $\gamma \approx 3/2$





DNS data from P. Schlatter, KTH

Outer flow approximation:

$$S^+ = \frac{dU^+}{dy^+} = \sqrt{W^+} / \ell_M^+ \approx \sqrt{\tau^+} / \ell_M^+$$

integration

Mean velocity defect: $U_d^+ = U_c^+ - U^+(r) \approx \frac{1}{\kappa} \int_0^r \frac{\sqrt{\tau^+}}{\ell_M^+} dr$

A unified velocity-defect law :

$$U^+(r) = U_c^+ - \frac{1}{\kappa} f(r)$$

$$f^{Pipe}(r; r_c) = 5/(1+r_c^2)^{1/4} \int_0^r r'/[(1-r'^5)(r'^2+r_c^2)^{1/4}]dr'$$

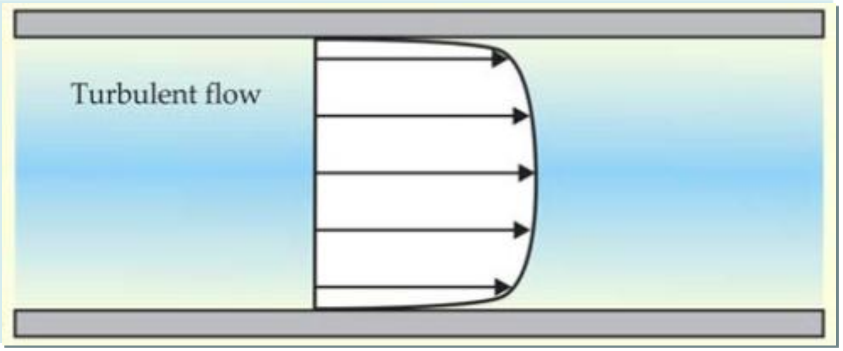
$$f^{CH}(r; r_c) = 4/(1+r_c^2)^{1/4} \int_0^r r'/[(1-r'^4)(r'^2+r_c^2)^{1/4}]dr'$$

$$f^{TBL}(r; \alpha) = 4 \int_{1-1/\alpha}^r \sqrt{1-[\alpha(1-r')]^{3/2}} / (1-r'^4) dr'$$

TBL

$U_c^+ = U_{99}^+$

Result 1： Mean velocity profile (MVP) - Pipe



Princeton pipe data（2004）

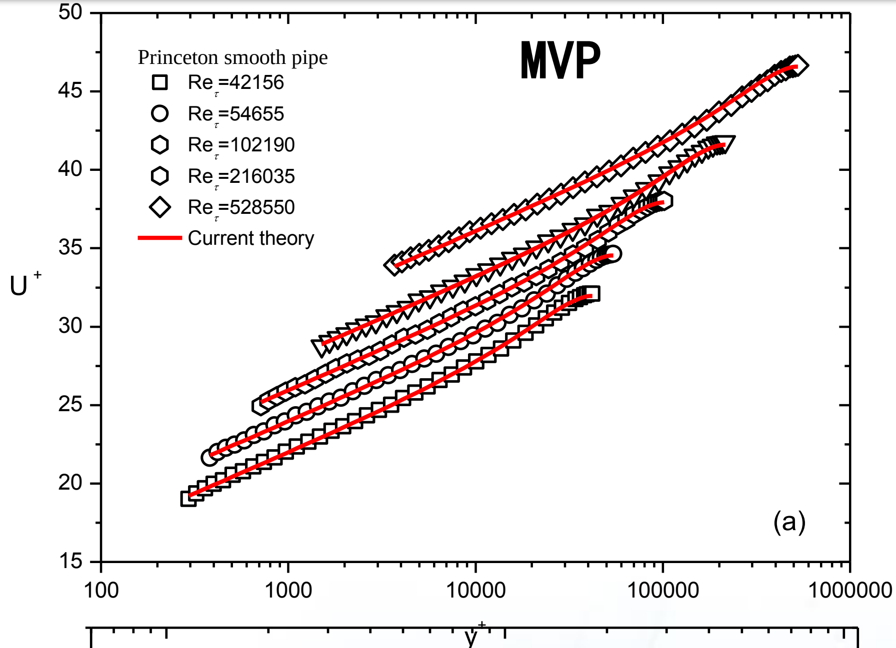
$$U^+(r) = U_c^+ - \frac{1}{\kappa} f(r, r_c)$$

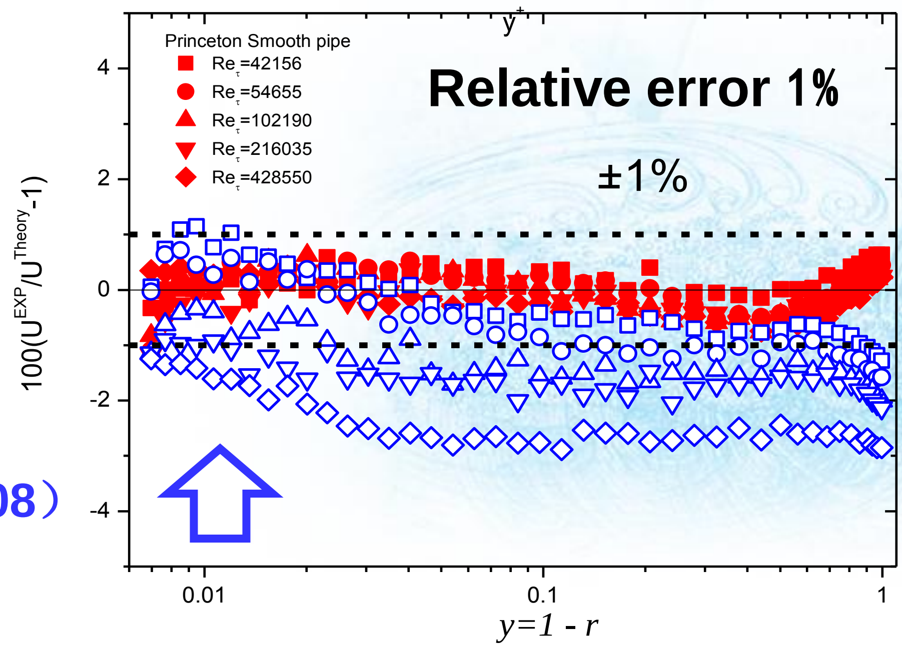
$$\kappa = 0.45$$

$$r_c = 0.50$$

r_c is insensitive to MVP

L'vov et al（PRL,2008）





Result 2：Mean velocity profile (MVP) - Channel

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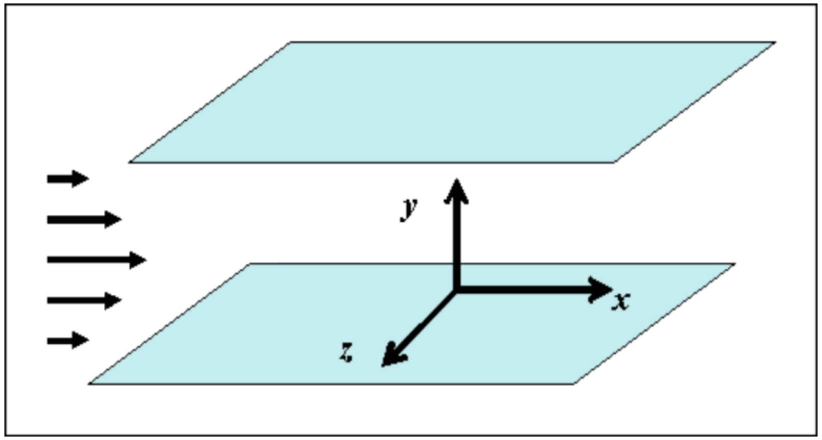


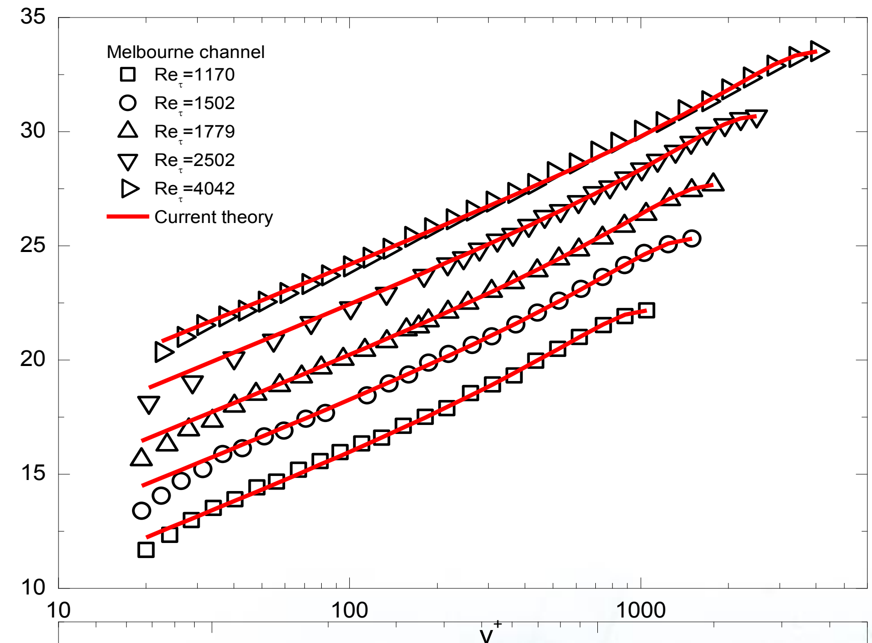
Melbourne Channel data
(Monty, 2005)

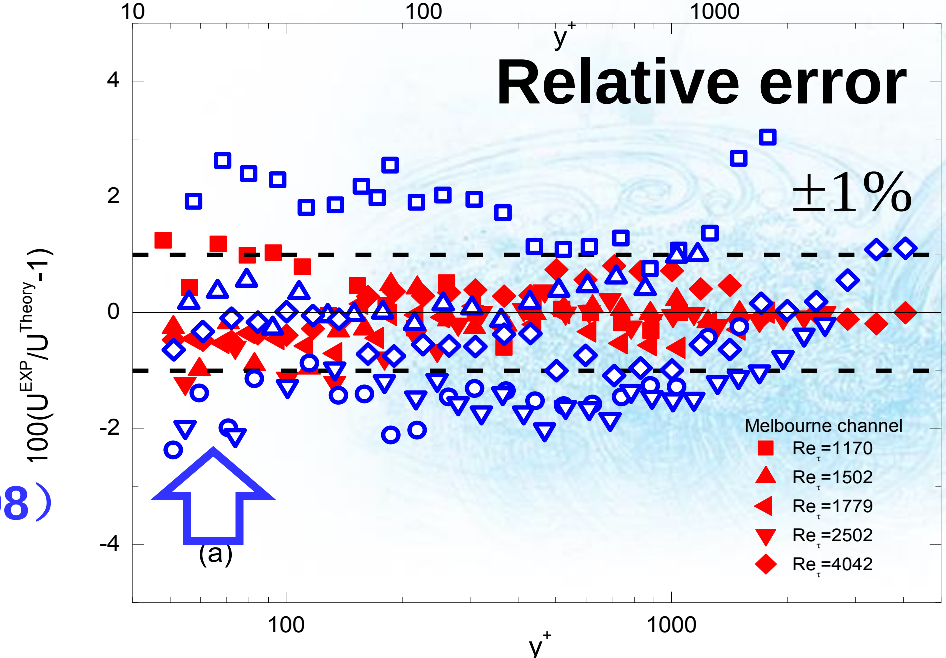
$$U^+(r) = U_c^+ - \frac{1}{\kappa} f(r, r_c)$$

$$\kappa = 0.45$$

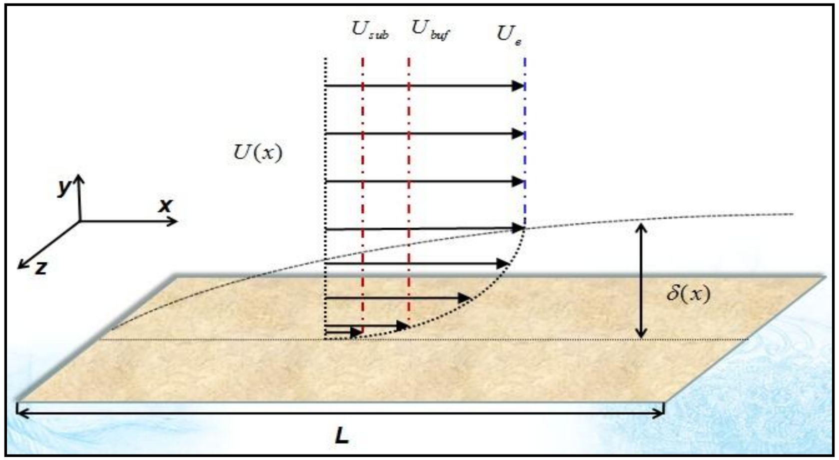
$$r_c = 0.37$$

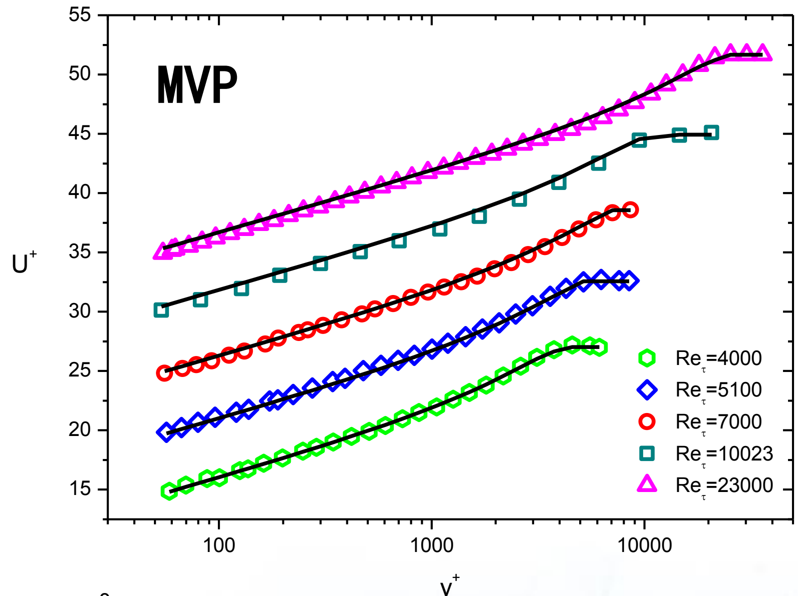
L'vov et al（PRL,2008）





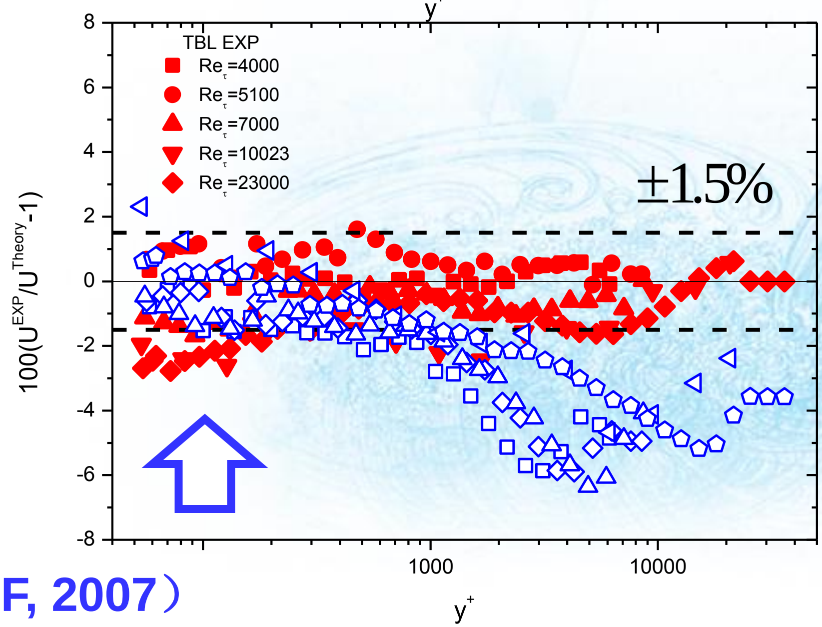
Result 3: Mean velocity profile (MVP) - TBL





TBL data（2000-2009）

EXP data:
Retau=4000 (Carlier etc.)
Retau=5100 (Carlier etc.)
Retau=7000 (Carlier etc.)
Retau=10023 (Degraff etc.)
Retau=23000 (Nickels etc.)



$$U^+(r) = U_c^+ - \frac{1}{\kappa} f(r, \alpha)$$

$$\kappa = 0.45 \qquad \alpha = 1$$

MCN（PoF, 2007）

Result 4： Mean velocity profile － Rough Pipe

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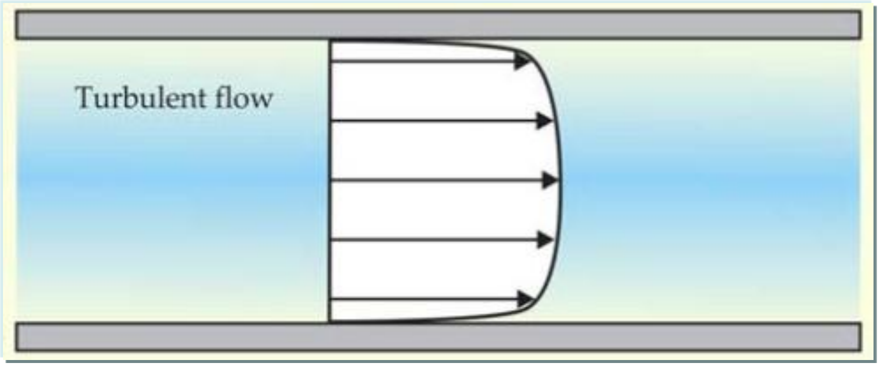


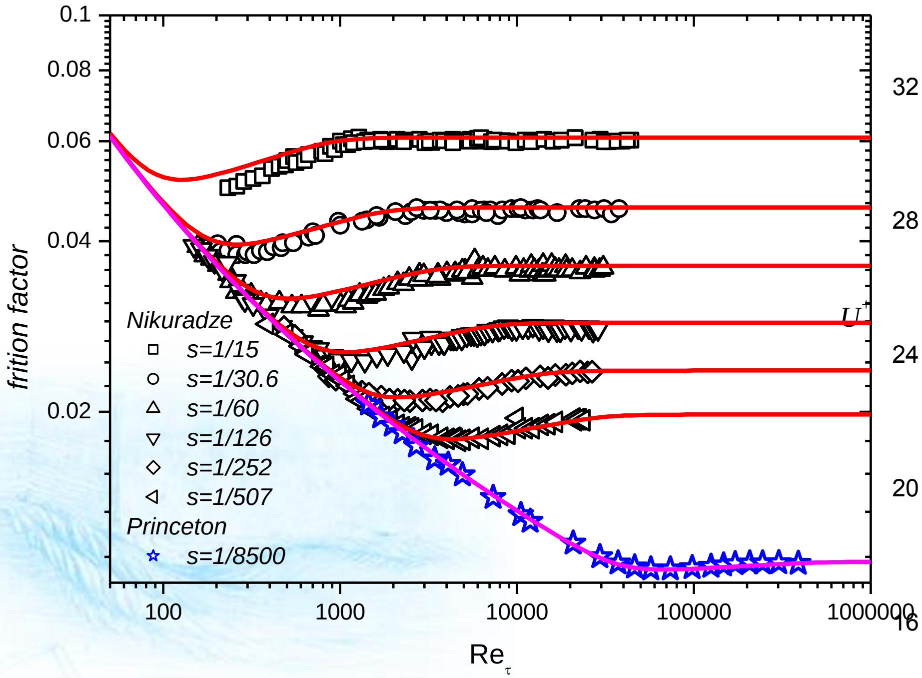
$$U^+(r) = U_c^+ - \frac{1}{\kappa} f(r, r_c)$$

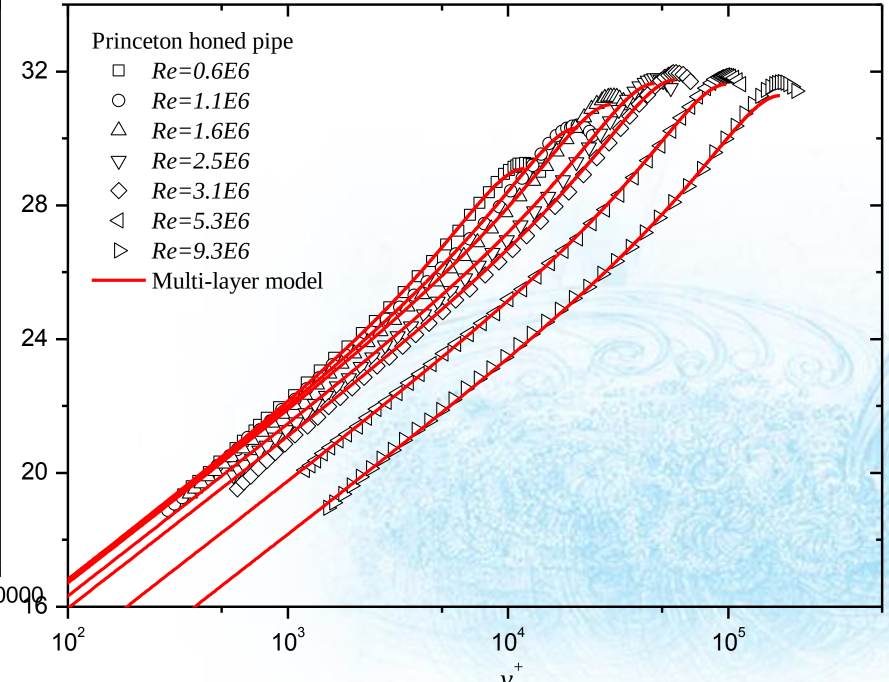
$$\kappa = 0.45$$

$$r_c = 0.50$$

Experiments（1933-2004）

r_c is insensitive to MVP





Relative error 1%

She, Wu, Chen, Hussain, New Journal of Physics, 2012

Conclusion: a unified bulk solution

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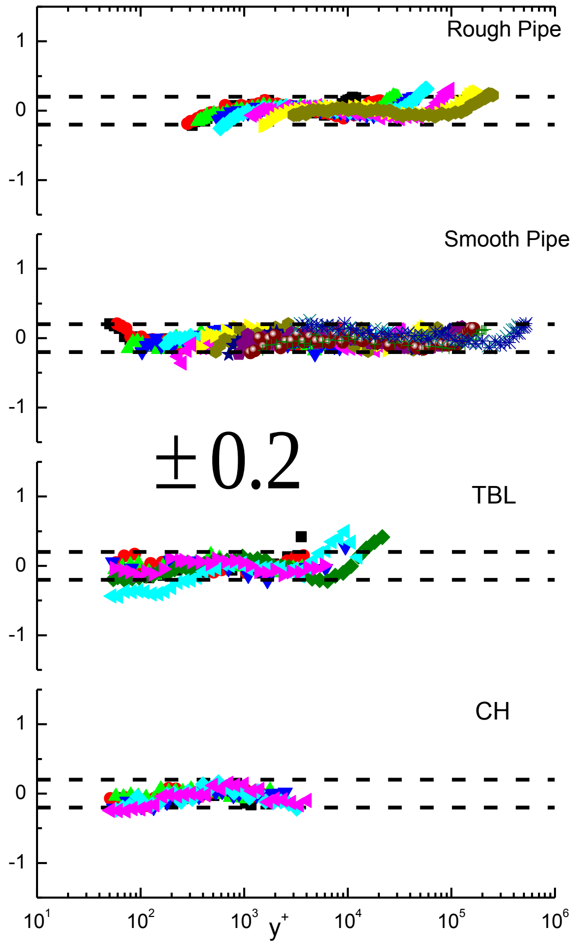
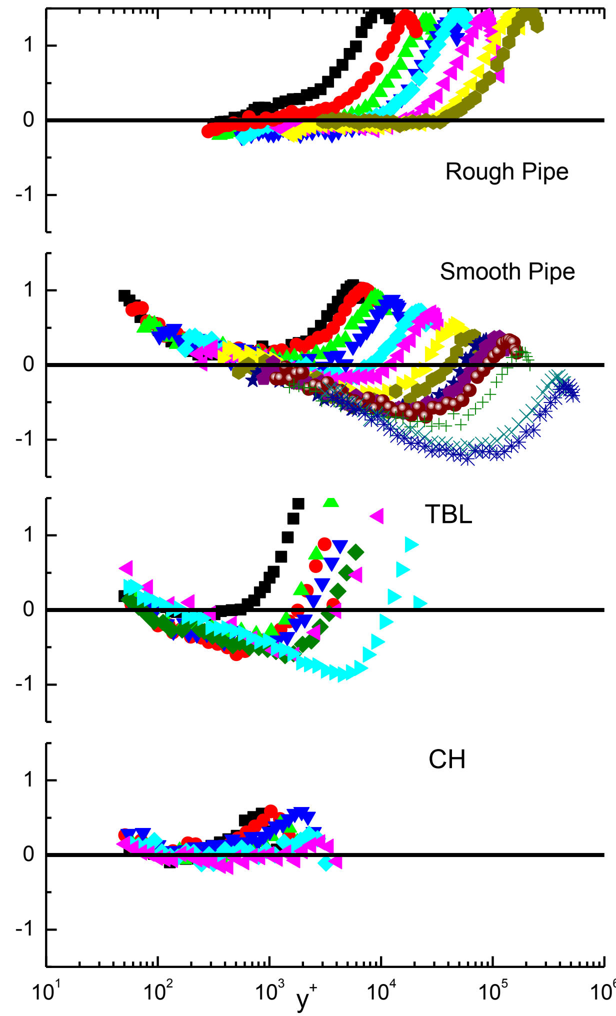
$$U^+ - (f/\kappa + U_c^+) \quad \kappa = 0.45$$



$$U^+ - (\ln y^+/\kappa + B)$$



$\kappa = 0.436$

Zagarola et al,1997

$\kappa = 0.391$

Marusic et al,2012

$\kappa = 0.38$

Alfredson et al, 2012

$\kappa = 0.387$

Marusic et al,2012

THANK YOU GRAZIE MERCI DANKE GRAZIAS 謝謝 СПАСИБО
GRACIAS OBRIGADO ありがとう DANK TAKK BEDANKT DAKUJEM