Measuring the joint probability density function of velocity and higher moments of the velocity fluctuations across a turbulent boundary layer

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Mathematical Description of 3C-3D JPDF

(Soria, J. & Willert, C. (2012) On measuring the joint probability density function of three-dimensional velocity components in turbulent flows. MST.)

- joint probability density function (JPDF) of 3C velocity components in 3D and time:

$$B_{u_1\,u_2\,u_3}(u_{1_0}, u_{2_0}, u_{3_0}; x_1, x_2, x_3, t)$$

  - sufficient to describe the statistical nature of turbulent flows in full detail

$$\begin{aligned} Prob\{u_{1_0} < u_1 < u_{1_0} + du_{1_0}, u_{2_0} < u_2 < u_{2_0} + du_{2_0}, \\ u_{3_0} < u_3 < u_{3_0} + du_{3_0}\}(x_1, x_2, x_3, t) \\ = B_{u_1\,u_2\,u_3}(u_{1_0}, u_{2_0}, u_{3_0}; x_1, x_2, x_3, t)\, du_{1_0}\, du_{2_0}\, du_{3_0} \end{aligned}$$

- all statistical moments can be computed once JPDF is known:

$$E[u_1^r\, u_2^n\, u_3^m] = \int_{-\infty}^{\infty} u_{1_0}^r\, u_{2_0}^n\, u_{3_0}^m\; B_{u_1\,u_2\,u_3}(u_{1_0}, u_{2_0}, u_{3_0}; x_1, x_2, x_3, t)\, du_{1_0}\, du_{2_0}\, du_{3_0}$$

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- shorthand notation used for the statistically stationary JPDF of the 3D velocity components:

$$p_{\mathbf{\Delta x}}(\mathbf{\Delta x}; \mathbf{x}) \quad = B_{\Delta x_1\,\Delta x_2\,\Delta x_3}(\Delta x_{1_0}, \Delta x_{2_0}, \Delta x_{3_0}; x_1, x_2, x_3)$$

$$\text{where } \mathbf{\Delta x} = \mathbf{\Delta u}\,\Delta t$$

Mathematical Formulation of 3C-3D JPDF

- 3C-3D Cross-correlation Function of Single Exposed Interrogation Volume Pairs Containing *N* Tracer Particles (Soria, J. (2006) Lecture Notes on Turbulence and Coherent Structures in Fluids, Plasmas and Nonlinear Media. pp 309–348, World Scientific.)

$$R(\eta) \quad = \sum_{i=1}^{N} R_{ii}\left(\eta - \boldsymbol{\Delta}\mathbf{x}_i\right) + \sum_{\substack{i=1,j=1 \\ i\neq j}}^{N} R_{ij}\left(\eta - \left(\mathbf{x}_j - \mathbf{x}_i + \boldsymbol{\Delta}\mathbf{x}_j\right)\right)$$

where

$$R_{ii}(\eta) \equiv \mathcal{F}^{-1}[G_{ii}(\mathbf{f})] = \int_{\Omega_{\mathbf{x}_l}} I_i(\mathbf{x},t) I_i(\mathbf{x}+\eta, t+\Delta t)\, d\mathbf{x}$$

$$R_{ij}(\eta) \equiv \mathcal{F}^{-1}[G_{ij}(\mathbf{f}))] = \int_{\Omega_{\mathbf{x}_l}} I_i(\mathbf{x},t) I_j(\mathbf{x}+\eta, t+\Delta t)\, d\mathbf{x}$$

$$G_{ij}(\mathbf{f}) = \mathcal{F}[I_i(\mathbf{x},t)]\ \mathcal{F}[I_j(\mathbf{x}+\eta, t+\Delta t)]^*$$

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Mathematical Formulation of 3C-3D JPDF

- 3C-3D Cross-correlation Function of Single Exposed Interrogation Volume *(IV)* Pairs Containing *N* Tracer Particles

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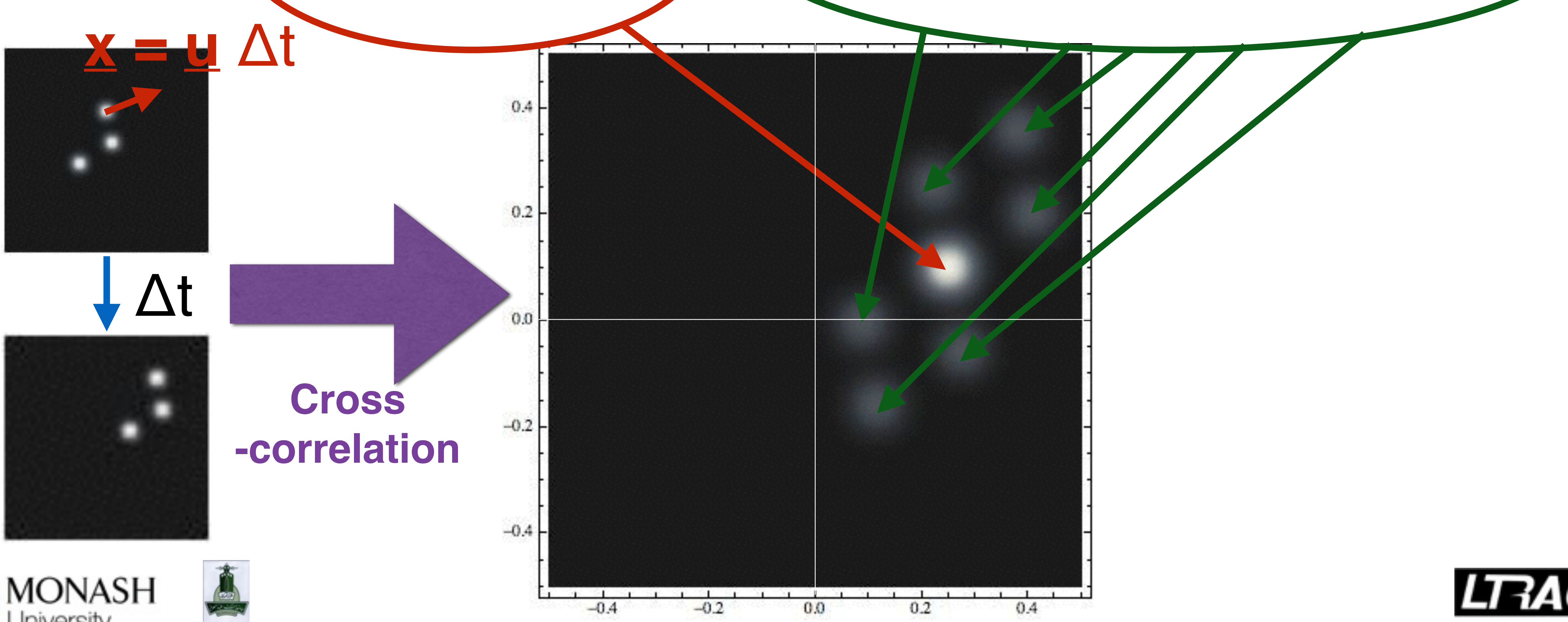
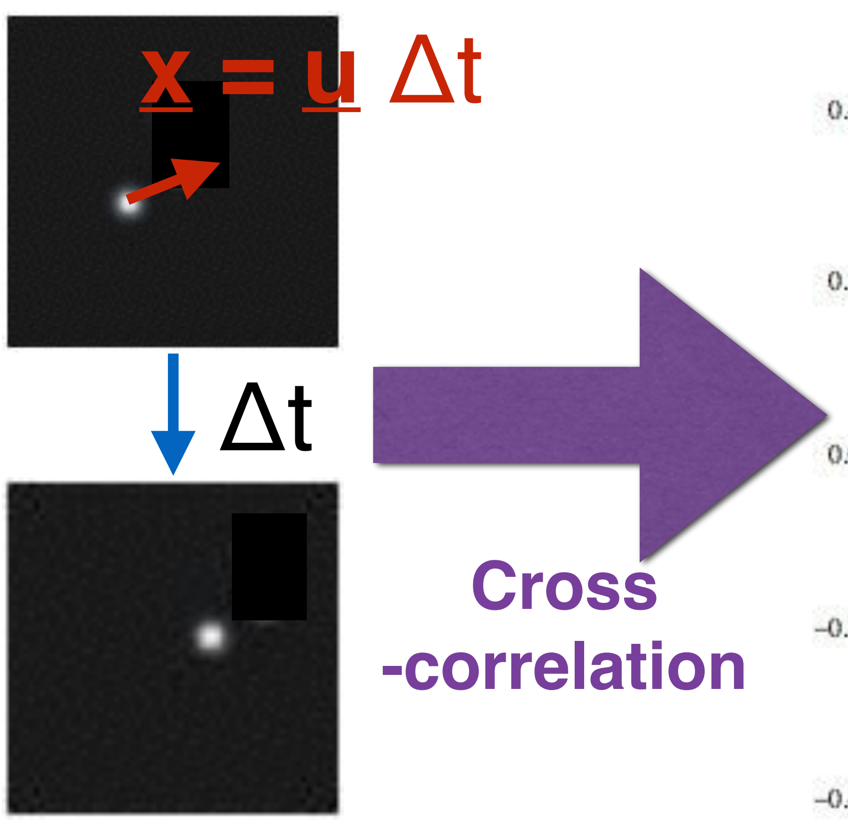
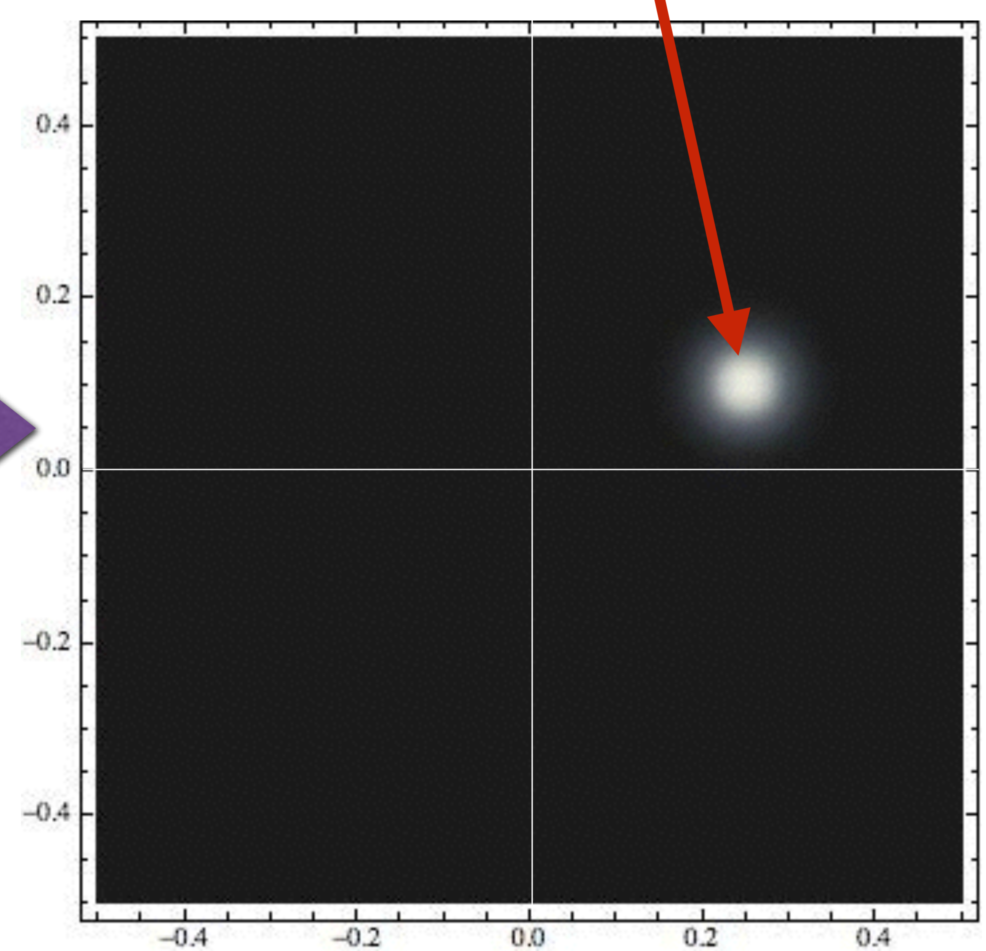


Mathematical Formulation of 3C-3D JPDF: From the Ensemble Average Cross-correlation Function to the Joint Probability Density Function

- assume that there is only one tracer particle within the interrogation volume
  - represents one sample of instantaneous velocity plus noise within the measurement volume
  - the 3D cross-correlation function is directly deduced to be:

$$R(\eta) = R_{ii}\left(\eta - \mathbf{\Delta x}_i\right)$$





Mathematical Formulation of 3C-3D JPDF: From the Ensemble Average Cross-correlation Function to the Joint Probability Density Function

- the ensemble averaged cross-correlation (EACC) function measured from *M* statistically independent samples is given by:

uniform PDF of the random variable describing the location of the tracer particle $\mathbf{x}_i$ within the spatial domain

$$
\begin{aligned}
E\left[R\left(\eta\right)\right] &= \lim_{M\to\infty} \frac{\sum_{k=1}^{M}\left(R(\eta)\right)_k}{M} \\
&= \int_{\Omega_{\boldsymbol{\Delta}\mathbf{x}_i}} \left[\int_{\Omega_{\mathbf{x}_l}} R_{\mathrm{ii}}\left(\eta - \boldsymbol{\Delta}\mathbf{x}_i\right) p_{\mathbf{x}_l}\left(\mathbf{x}_i\right) d\mathbf{x}_i\right] p_{\boldsymbol{\Delta}\mathbf{x}_i}\left(\boldsymbol{\Delta}\mathbf{x}_i\right) d\boldsymbol{\Delta}\mathbf{x}_i
\end{aligned}
$$

Mathematical Formulation of 3C-3D JPDF: From the Ensemble Average Cross-correlation Function to the Joint Probability Density Function

- this yields:

$$E\left[R\left(\eta\right)\right] = \int\limits_{\Omega_{\mathbf{\Delta x}_i}} R_{\mathrm{ii}}\left(\eta - \mathbf{\Delta x}_i\right)\; p_{\mathbf{\Delta x}_i}\left(\mathbf{\Delta x}_i\right) d\mathbf{\Delta x}_i .$$

- it can be shown via the convolution theorem that the 3C-3D JPDF is given as:

ensemble averaged single particle cross-correlation function

Computationally more efficient

$$p_{\mathbf{\Delta x}_{\mathrm{i}}}(\eta) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}\left[E\left[R(\eta)\right]\right]}{\mathcal{F}\left[R_{ii}(\eta)\right]}\right] = \mathcal{F}^{-1}\left[\frac{E\left[\mathcal{F}\left[R(\eta)\right]\right]}{\mathcal{F}\left[R_{ii}(\eta)\right]}\right]$$

Demonstration of Performance - 3C-3D JPDF

- Numerical simulations using 3D Gaussian velocity JPDF data with
  - Monte Carlo simulations with:
    - mean velocity components: (m_u, m_v, m_w),
    - standard deviations: $(\sigma_u, \sigma_v, \sigma_w)$ and
    - correlation coefficients: $\rho_{uv} \equiv \sigma_{uv}/(\sigma_u\, \sigma_v)$, $\rho_{uw} \equiv \sigma_{uw}/(\sigma_u\, \sigma_w)$ and $\rho_{vw} \equiv \sigma_{vw}/(\sigma_v\, \sigma_w)$
    - Gaussian particle intensity representing particles with diameter, d_i :

$$I_i(x,y,z) = I_{0_i} e^{\left[-\frac{18\,(x^2+y^2+z^2)}{d_i^2}\right]}$$

    - 10^6 - 3D particle volume samples are generated with:
      - $I_{0i} = I_0 = 1$ - same peak intensity for all particles
      - $d_i = d = 1, 2, 4, 6$
      - particle location within interrogation volume is given using a uniform 3D PDF

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Demonstration of Performance - 3C-3D JPDF

- without correction for particle size:

$$p_{\mathbf{\Delta x_i}}(\eta) = E\left[R(\eta)\right]$$

d	m_u	m_v	m_w	σ_u	σ_v	σ_w	ρ_{uv}	ρ_{uw}	ρ_{vw}
(input)	1.0	2.0	-3.0	3.0	1.5	1.0	0.5	0.1	-0.8
1.0	1.002	2.000	-3.001	3.020	1.541	1.064	0.4817	0.0944	-0.7310
2.0	1.002	2.000	-3.002	3.020	1.541	1.064	0.4817	0.0944	-0.7310
4.0	0.999	1.998	-3.003	3.141	1.768	1.371	0.4041	0.0699	-0.4944
6.0	0.998	1.997	-3.004	3.312	2.056	1.727	0.3295	0.0526	-0.3375

Demonstration of Performance - 3C-3D JPDF

- with correction for particle size:

$$p_{\mathbf{\Delta x}_i}(\eta) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}\left[E\left[R\left(\eta\right)\right]\right]}{\mathcal{F}\left[R_{\mathrm{ii}}\left(\eta\right)\right]}\right]$$

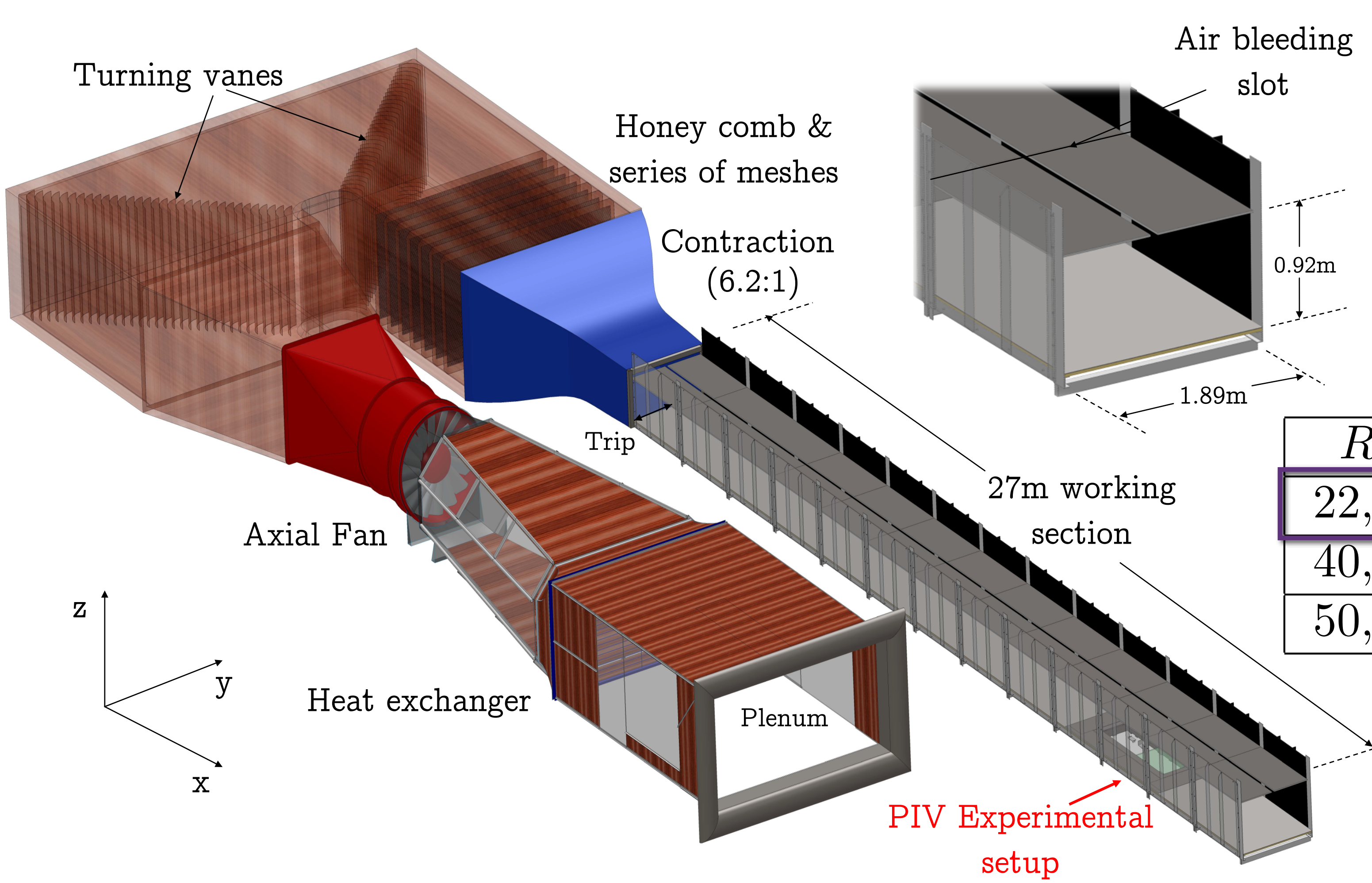
d	m_u	m_v	m_w	σ_u	σ_v	σ_w	ρ_{uv}	ρ_{uw}	ρ_{vw}
(input)	1.0	2.0	-3.0	3.0	1.5	1.0	0.5	0.1	-0.8
1.0	1.006	2.001	-3.000	3.010	1.505	1.013	0.4941	0.1029	-0.7834
2.0	1.002	2.000	-3.002	3.008	1.517	1.028	0.4914	0.0980	-0.7684
4.0	0.999	1.998	-3.003	2.998	1.499	0.999	0.4996	0.1005	-0.8000
6.0	0.998	1.997	-3.004	2.998	1.499	0.999	0.4997	0.1006	-0.8006

Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at Re_τ = 8000

(Collaborators: N.Buchmann, C. Atkinson, C.M. de Silva, E.P. Gnanamanickam, N. Hutchins, I. Marusic)

- experiments were undertaken in the high Reynolds number turbulent boundary layer wind tunnel at the Melbourne University
- measurements were taken at three different free-stream velocities

- de Silva *et al.* (2013) Nested multi-resolution PIV measurements of wall bounded turbulence at high reynolds numbers. PIV13, Delft, The Netherlands.
- Buchmann *et al.* (2013) Experimental investigation of the near and far field structure of high Reynolds number turbulent boundary layers. ETC14, Lyon, France.



Re_θ	Re_τ	U (m/s)	u_τ (m/s)	δ (m)
22,400	8,000	10	0.334	0.36
40,800	14,500	20	0.630	0.35
50,000	19,500	30	0.935	0.34

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Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at Re_τ = 8000

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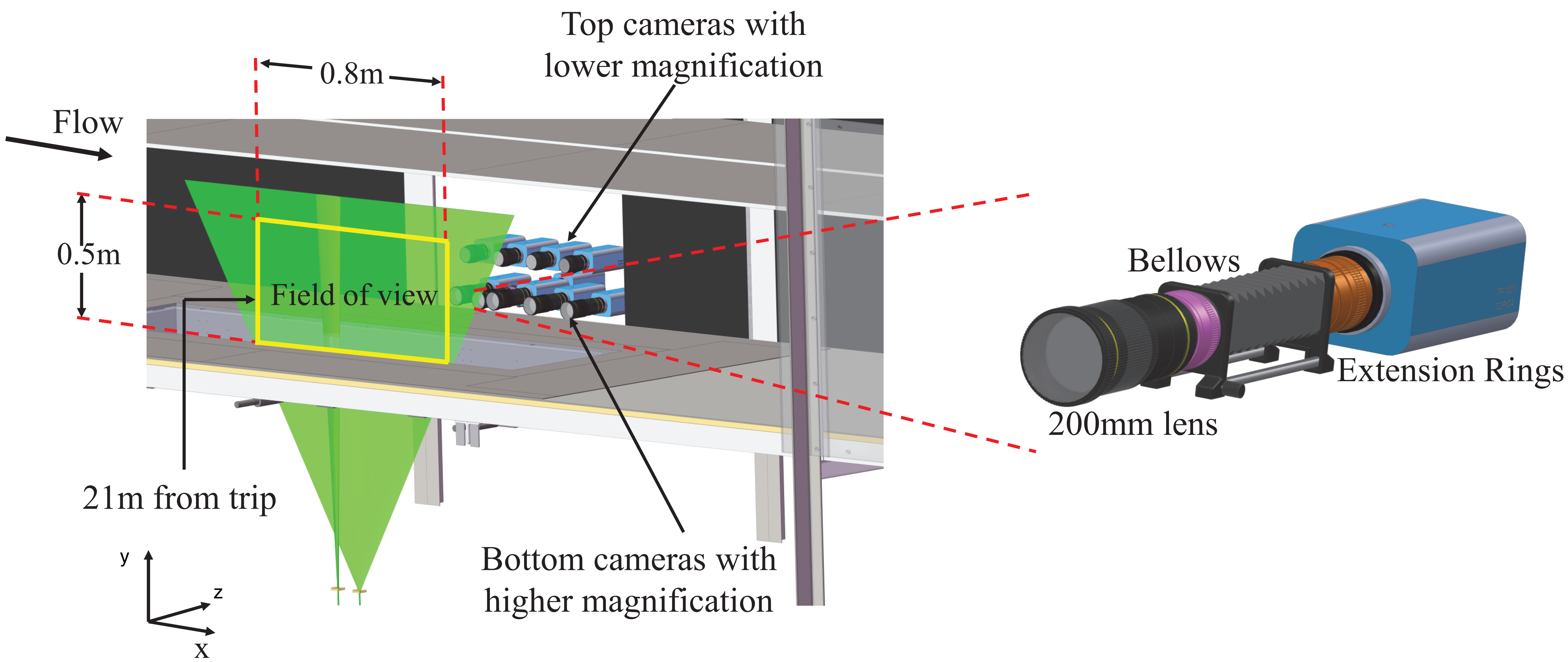
- PIV system consisting of an array of nine high resolution (4008 x 2672 px^2) cameras ~ total of 96 Mpx was used
- two dual cavity 400 mJ Nd:YAG lasers were used in order to use different interframe timing for the far field and the near wall cameras



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- Multi-resolution approach was used with the 9 cameras to capture a streamwise domain > 2δ and simultaneous resolve the near wall region

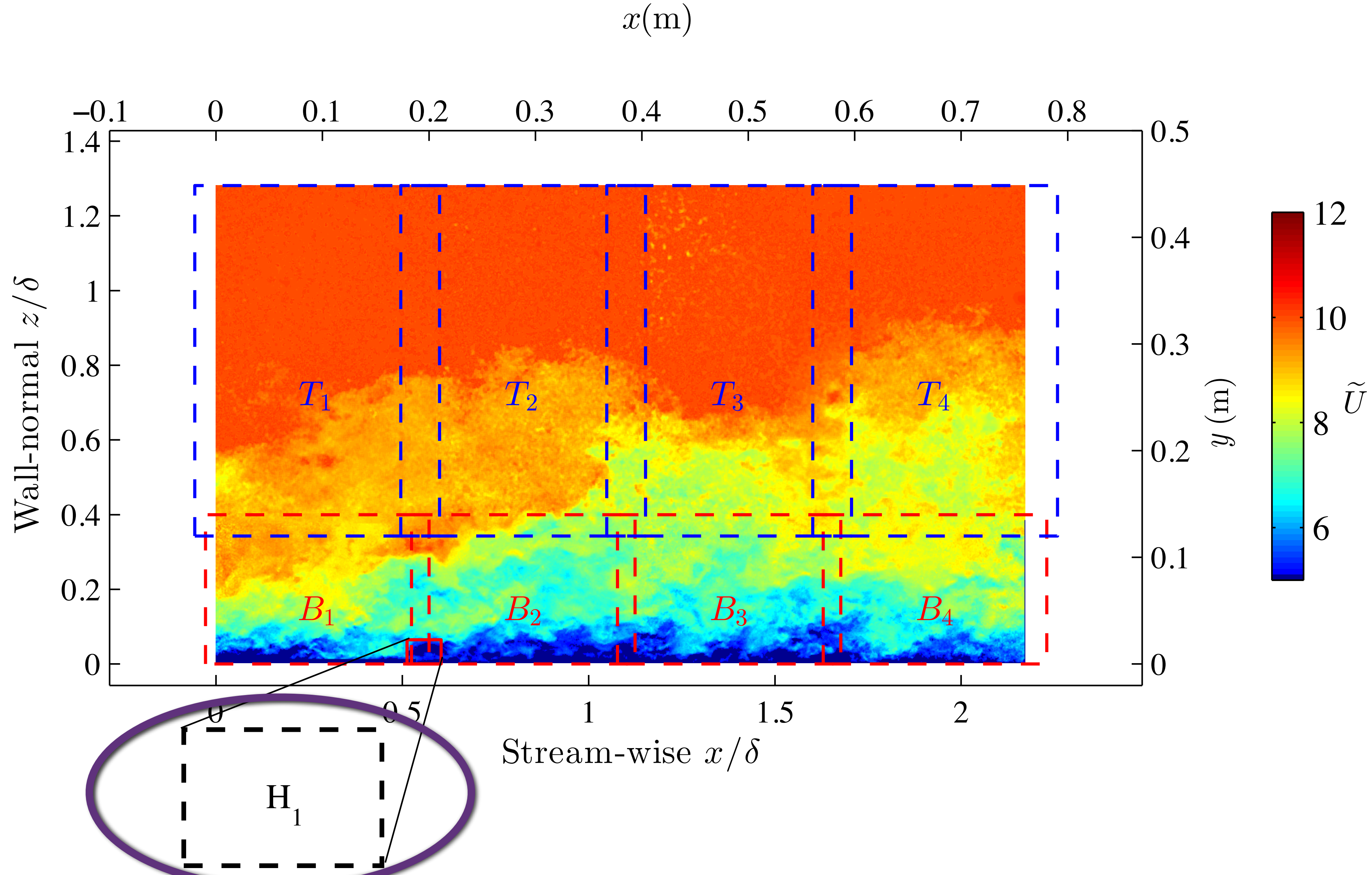
Spatial resolution

Δl^+

45^+ - 117^+

34^+ - 88^+

5^+ - 14^+



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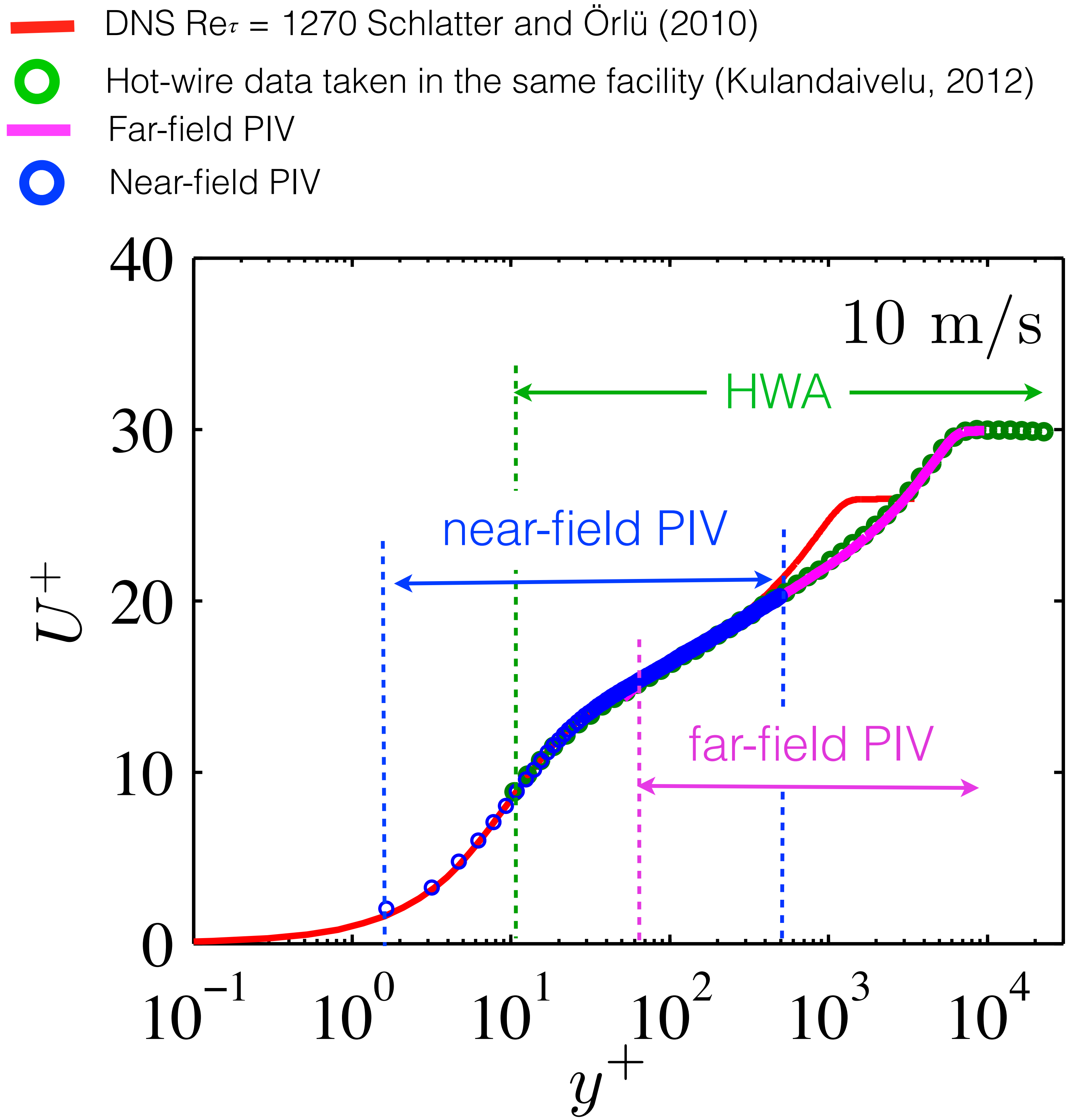

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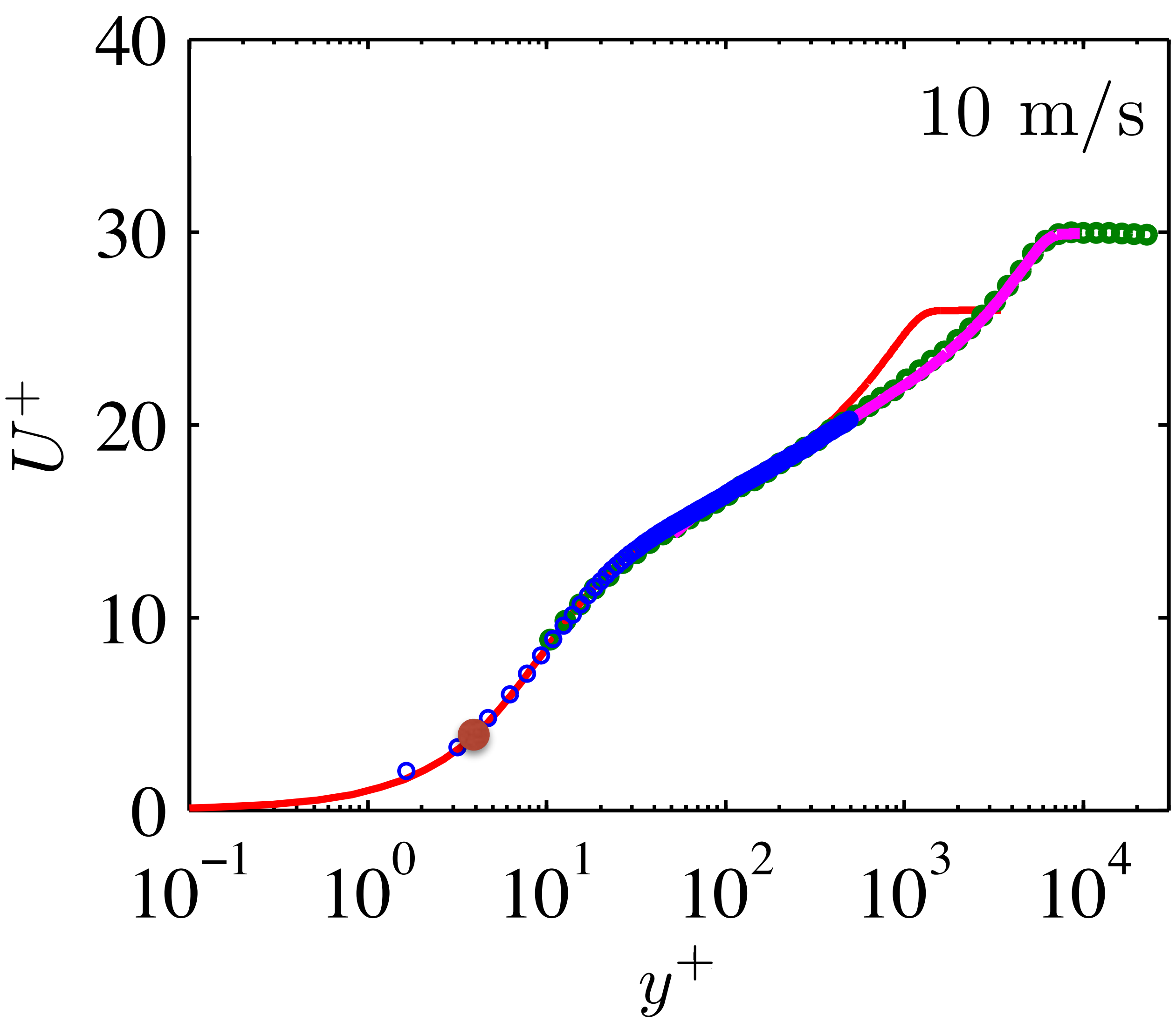
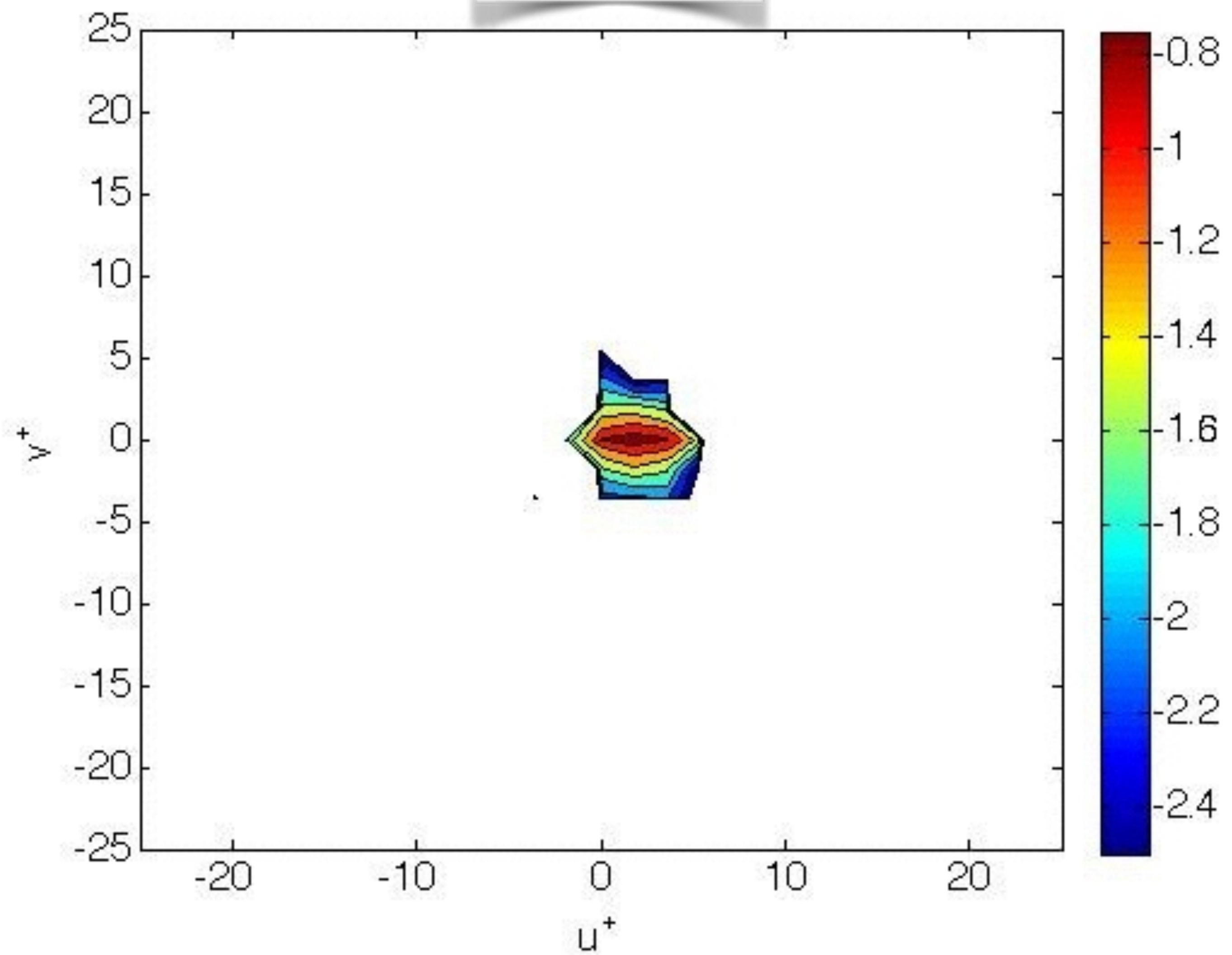
Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at $Re_\tau = 8000$

- 2C-2D JPDF:

$$IV = l_x^+ \times l_y^+ = 9.5 \times 1.2$$



$y^+ = 3$



Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at Re_τ = 8000

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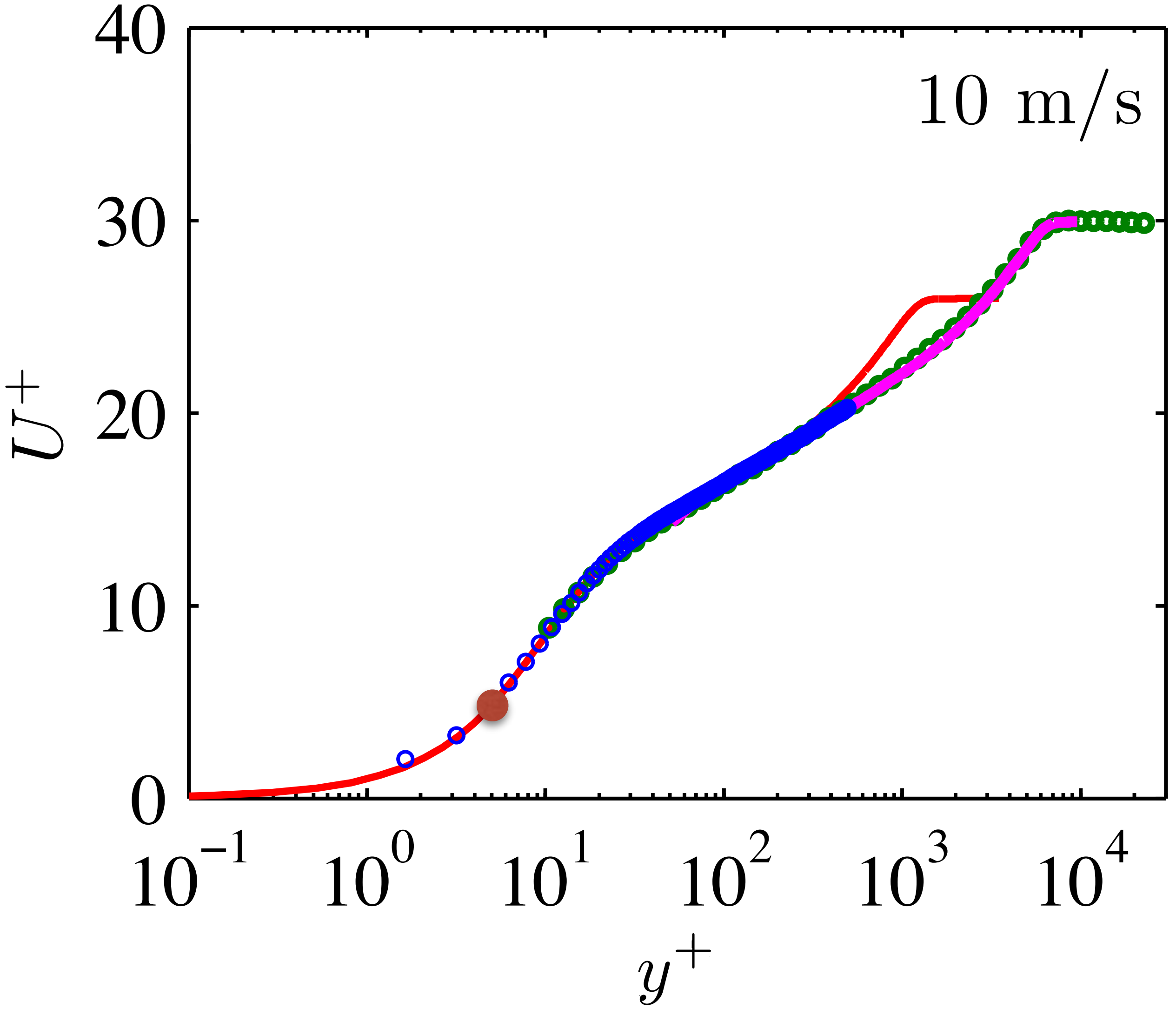
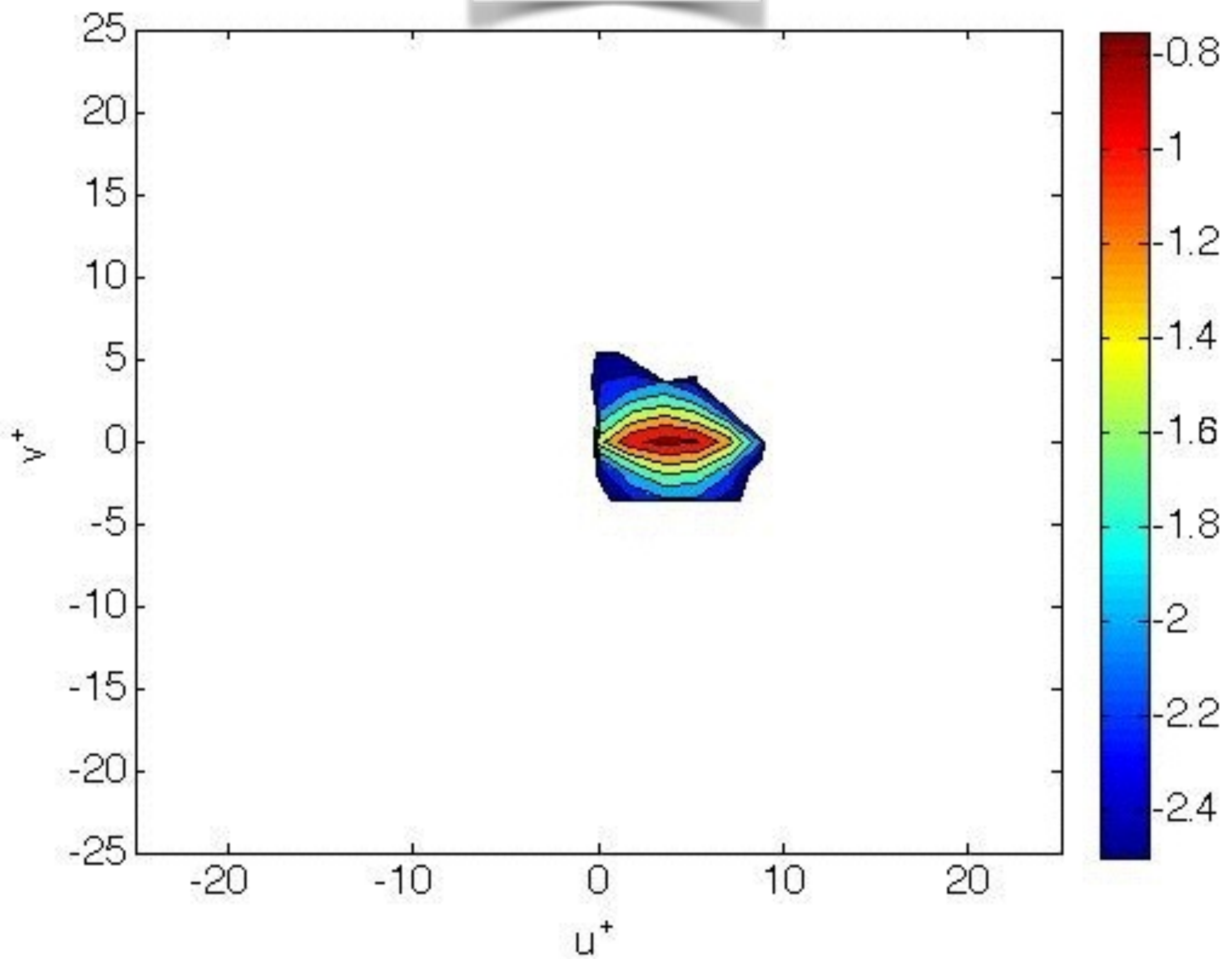


$y^+ = 5$



Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at Re_τ = 8000

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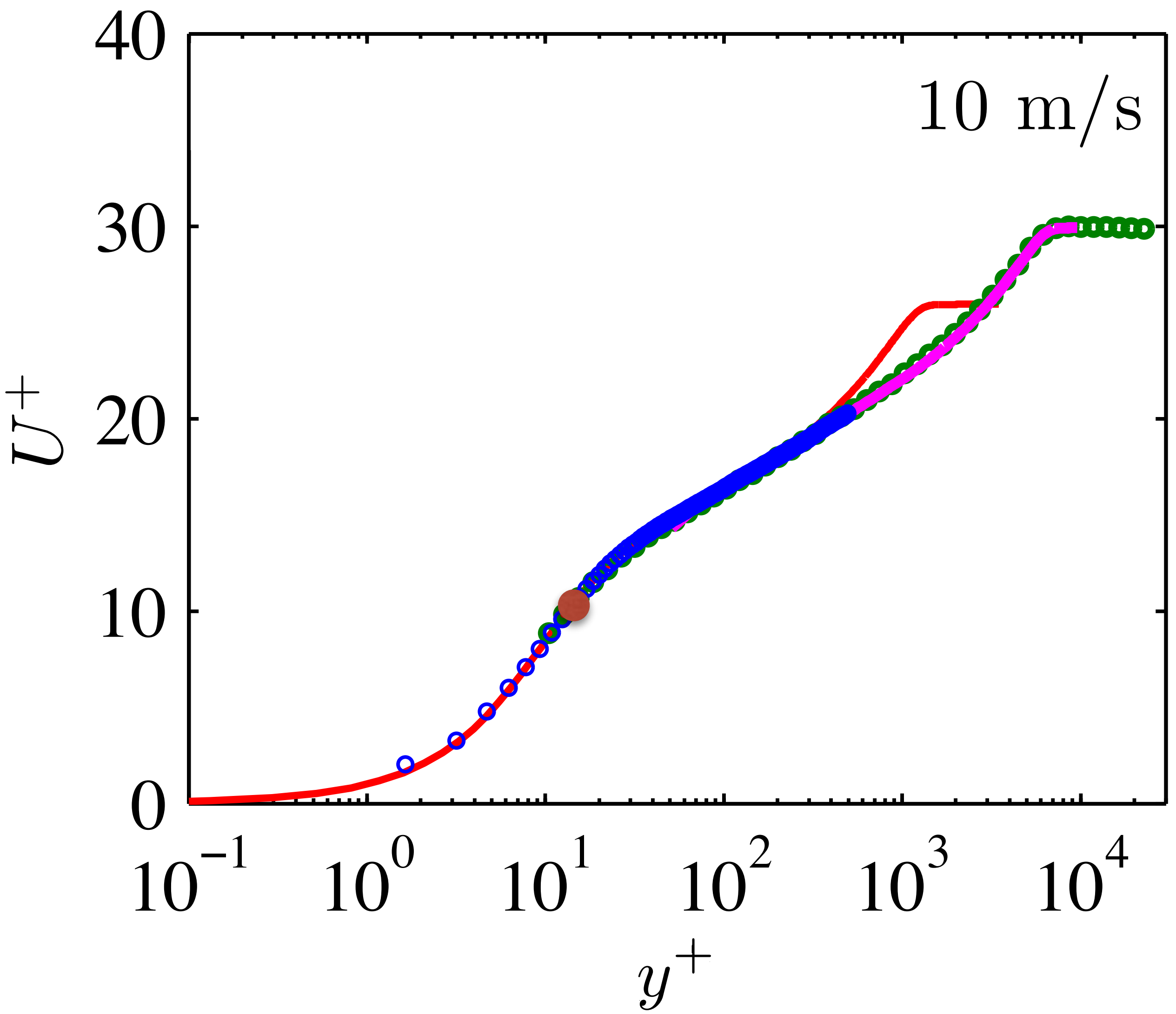
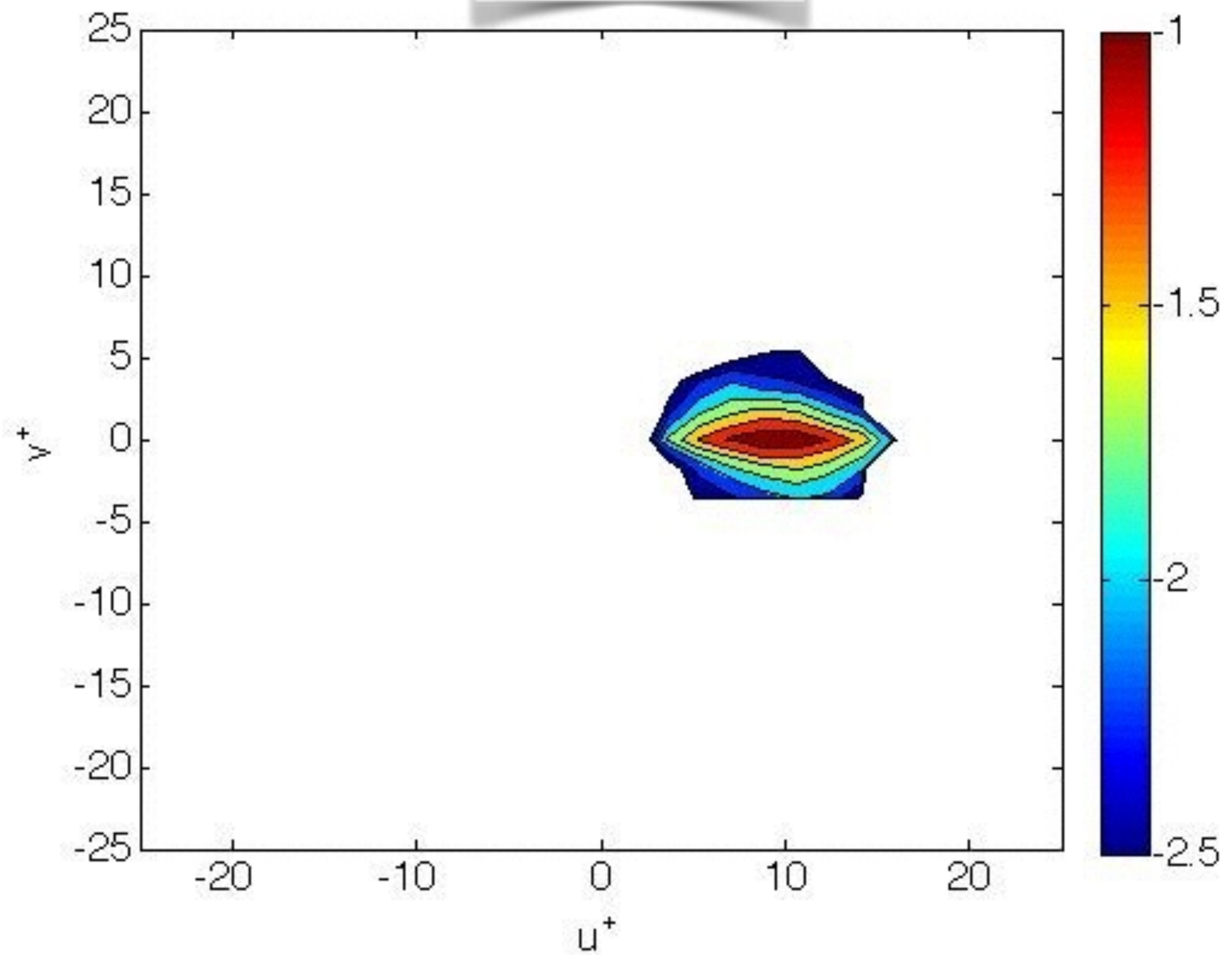


$y^+ = 15$



Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at $Re_\tau = 8000$

- 2C-2D JPDF:

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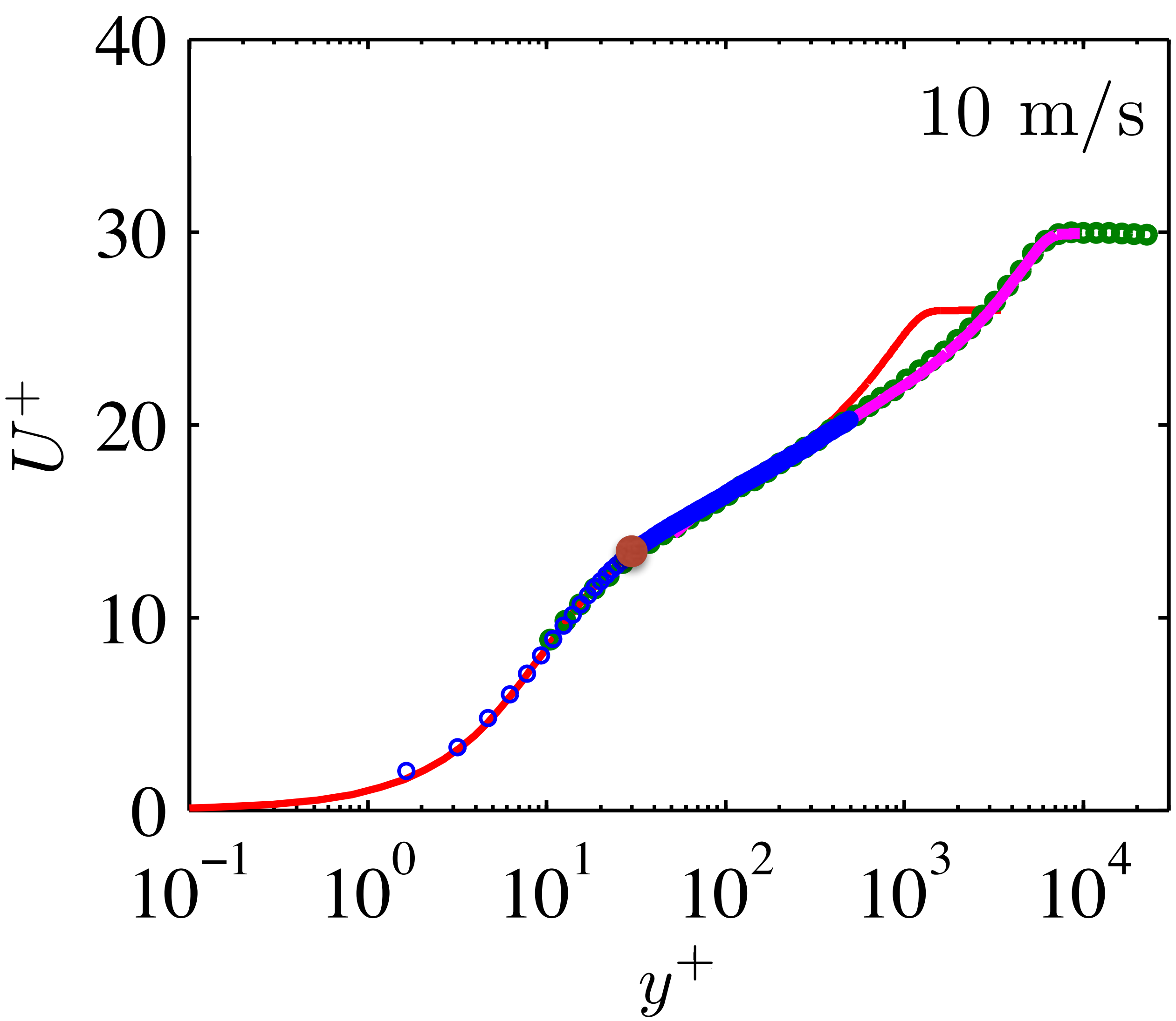

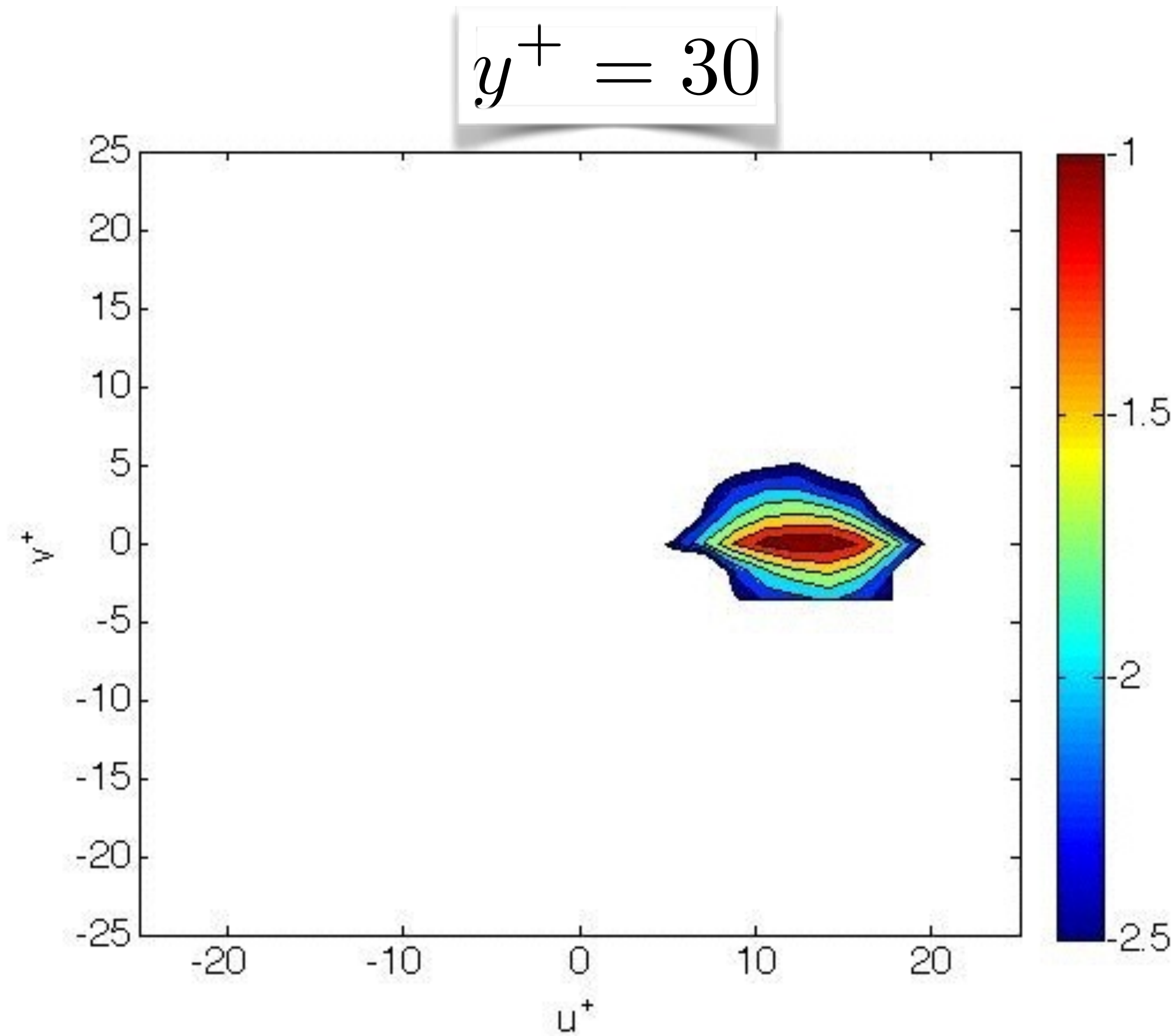

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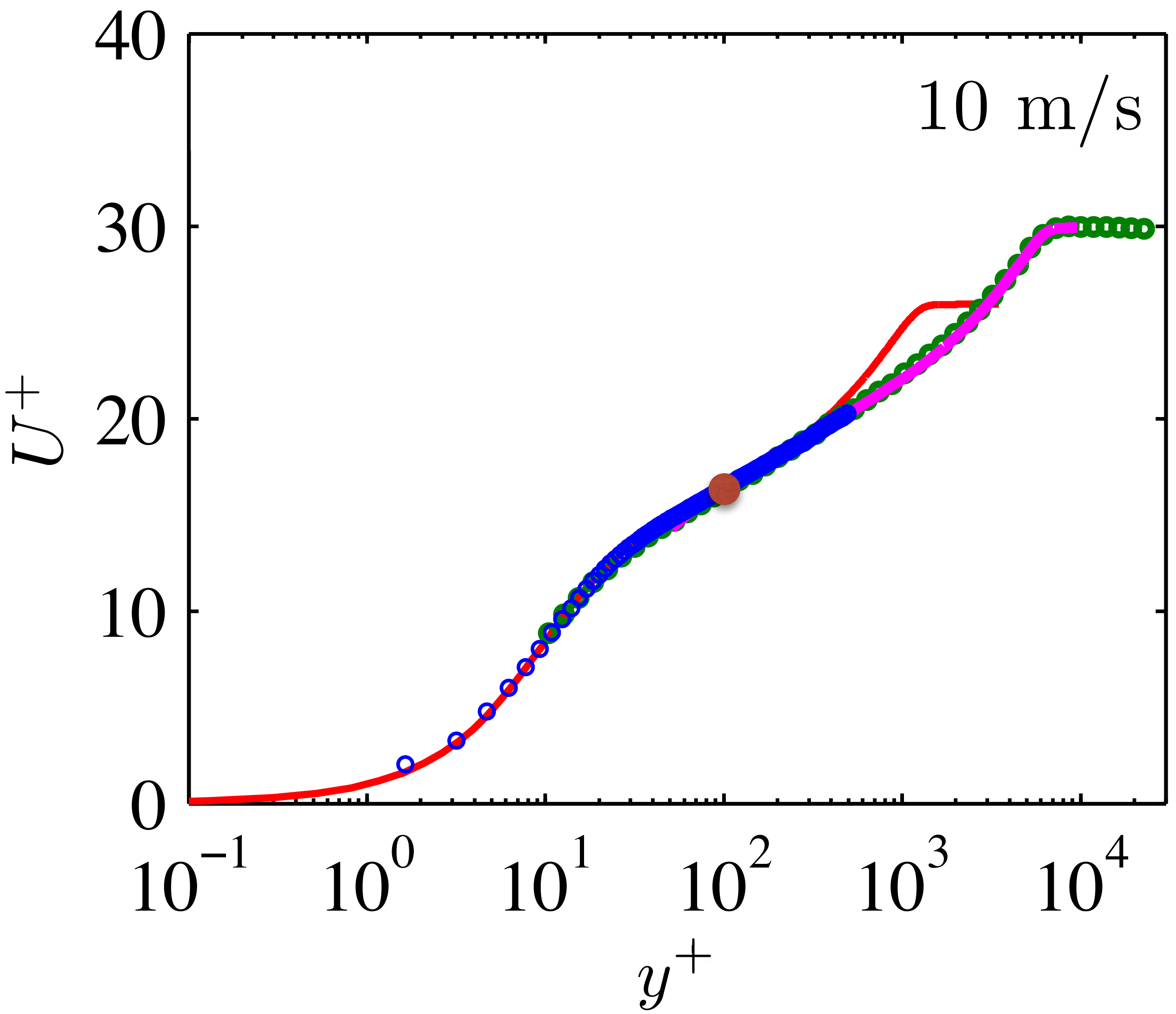
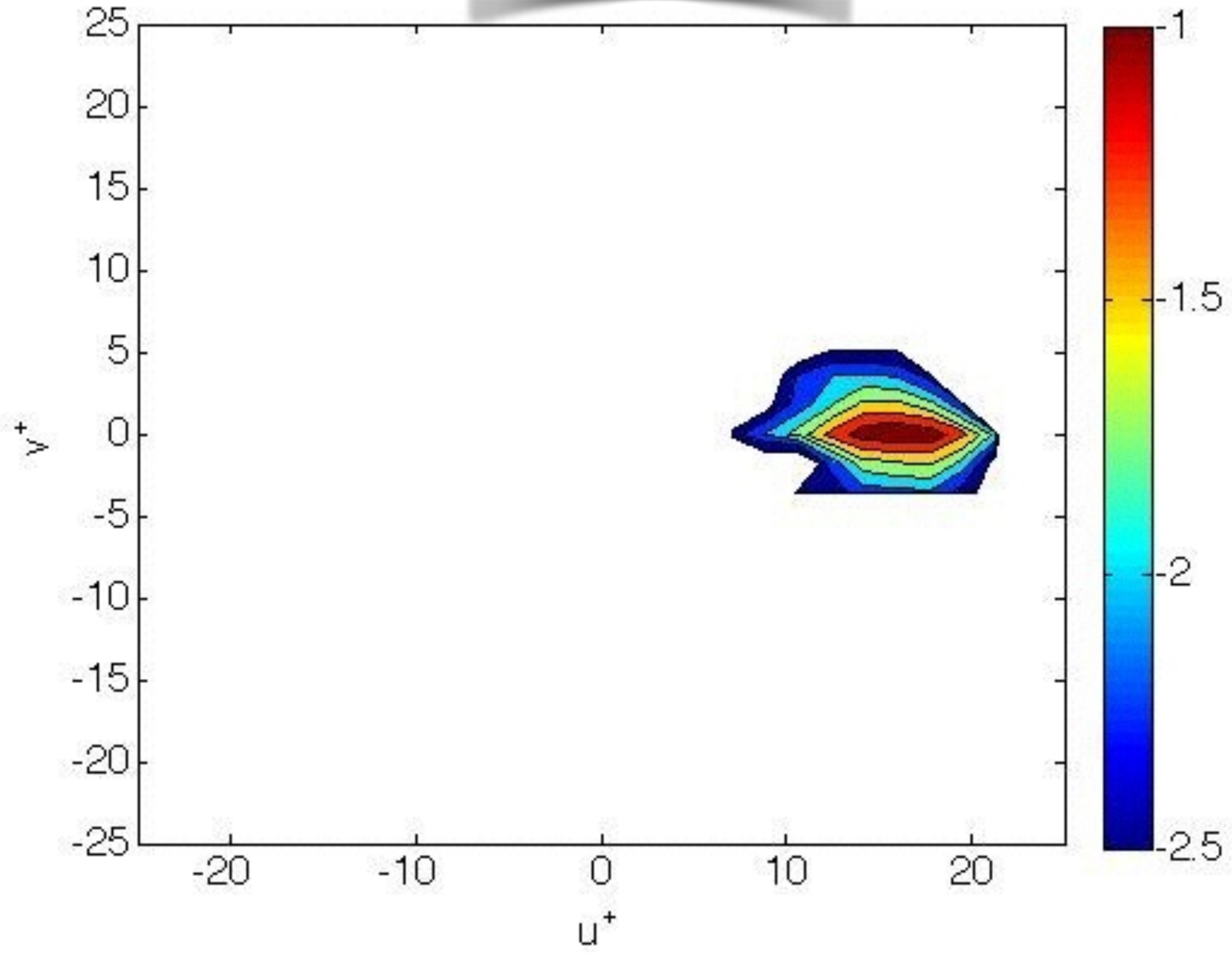


$y^+ = 100$



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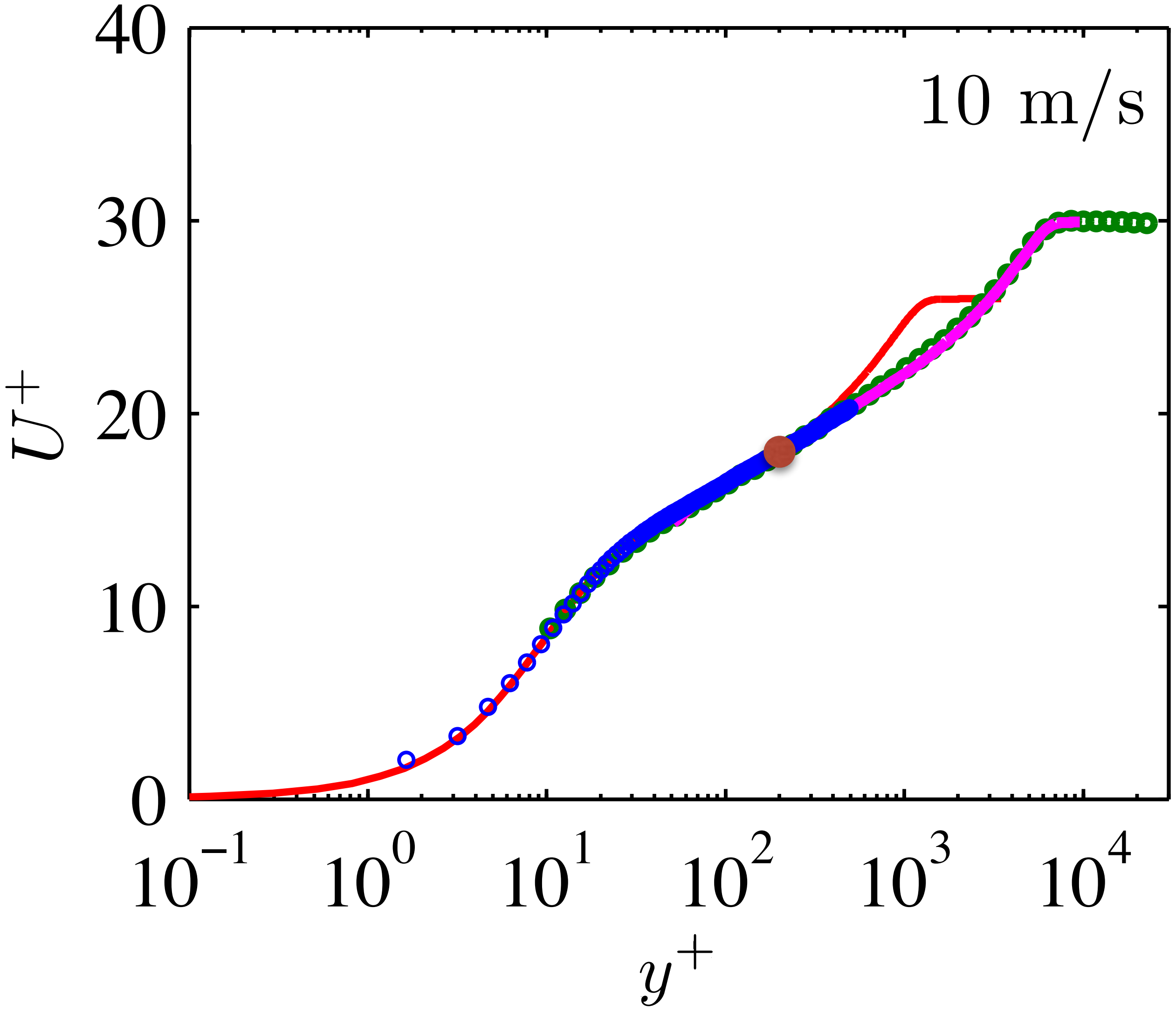

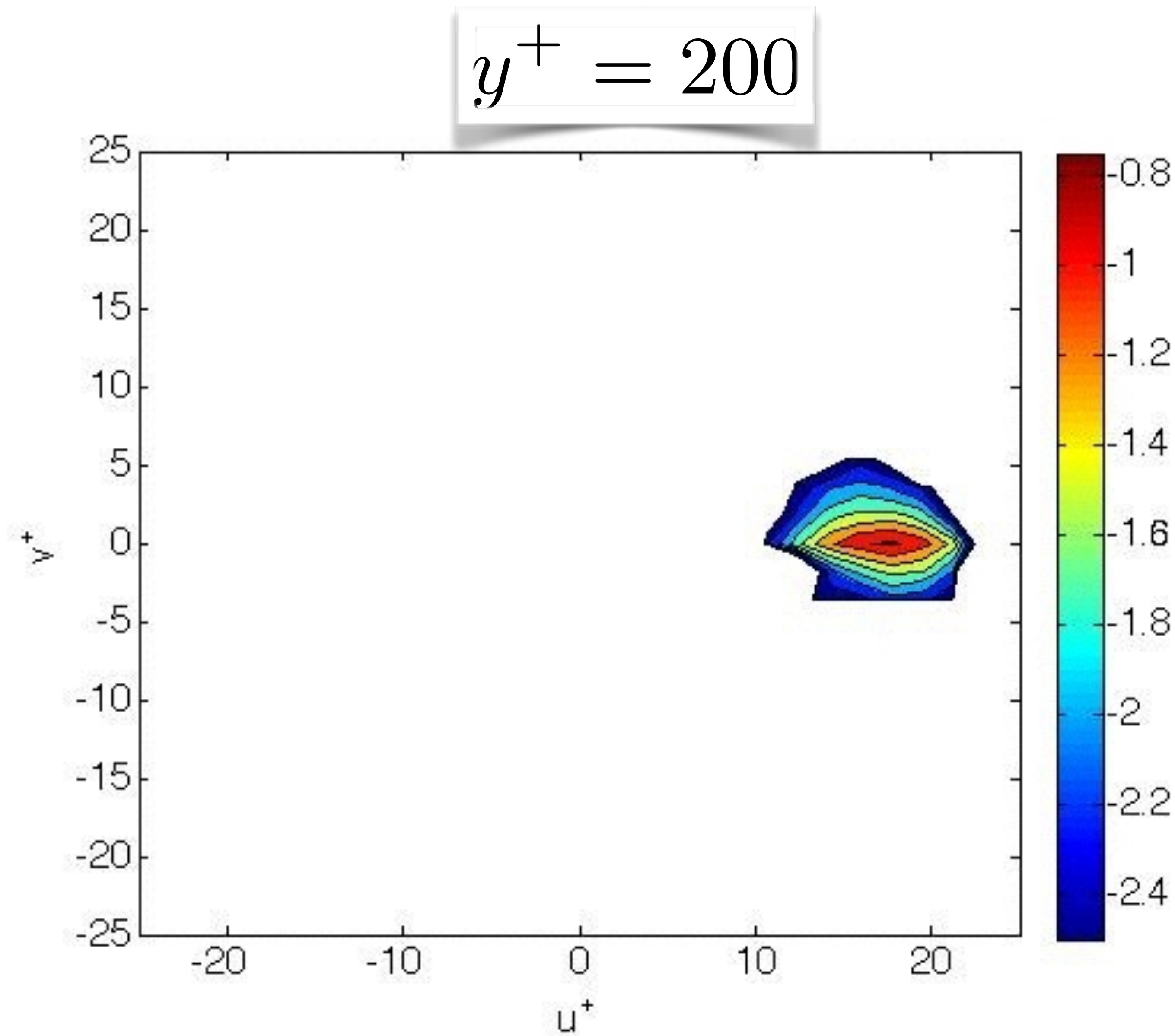

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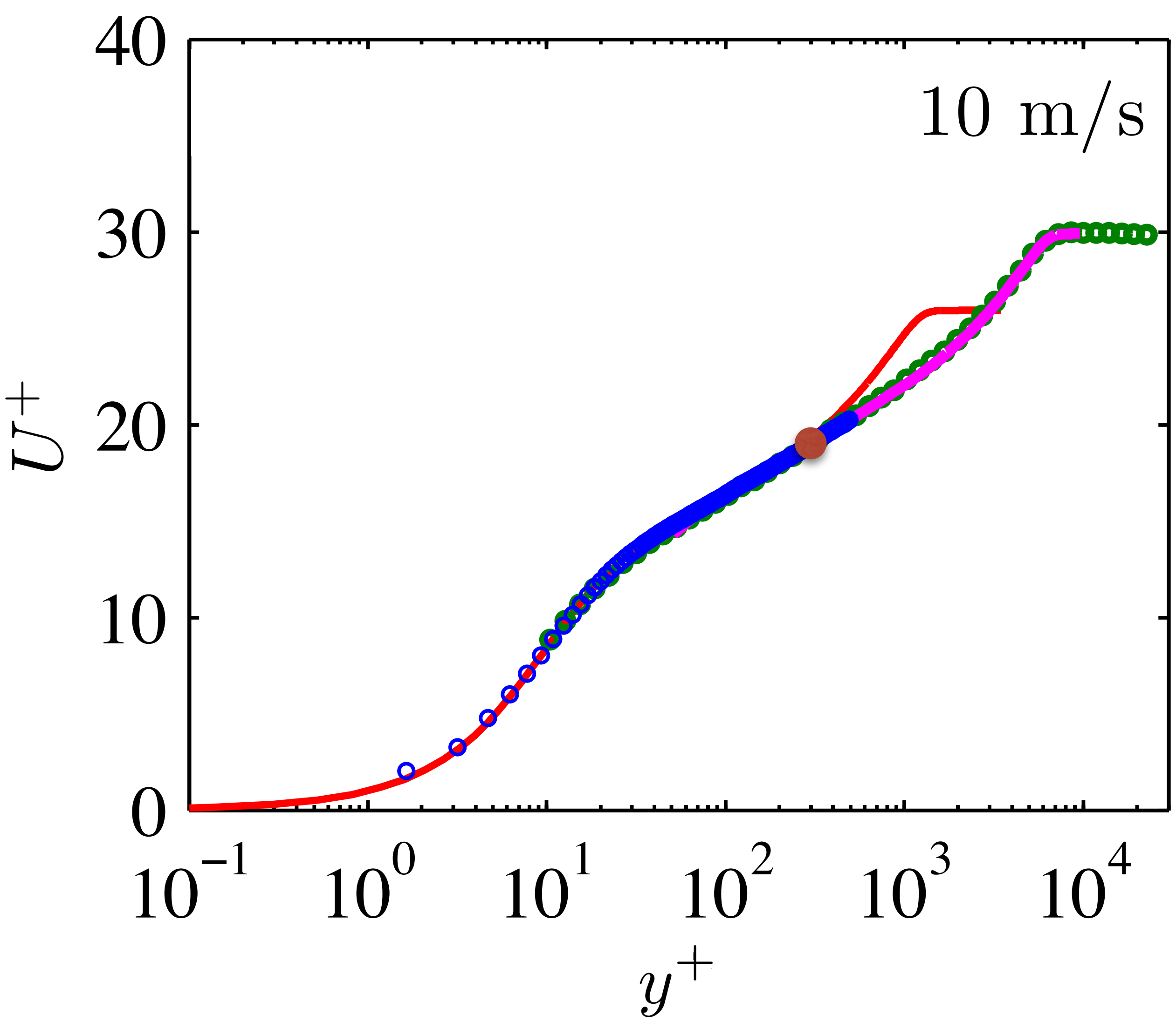
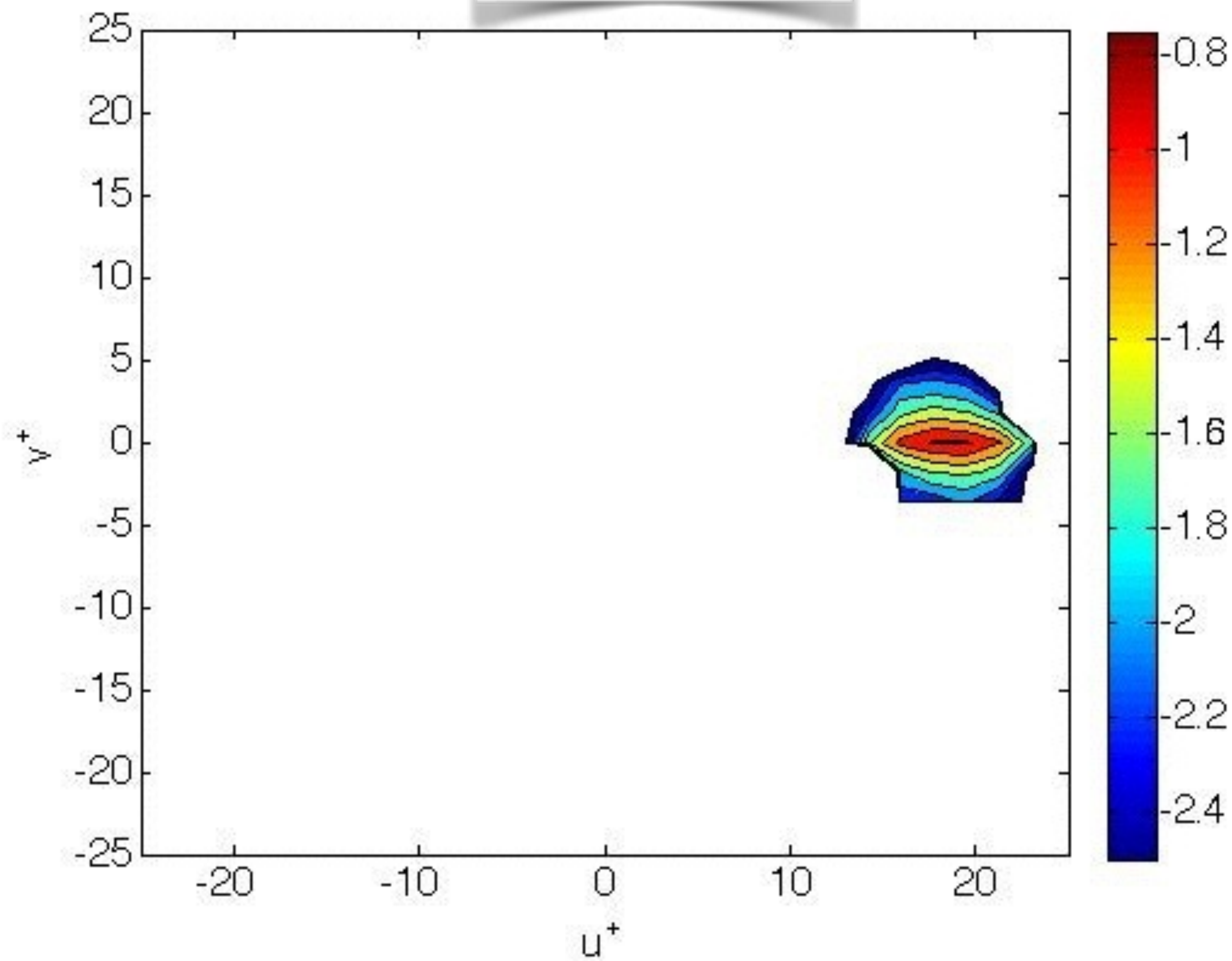


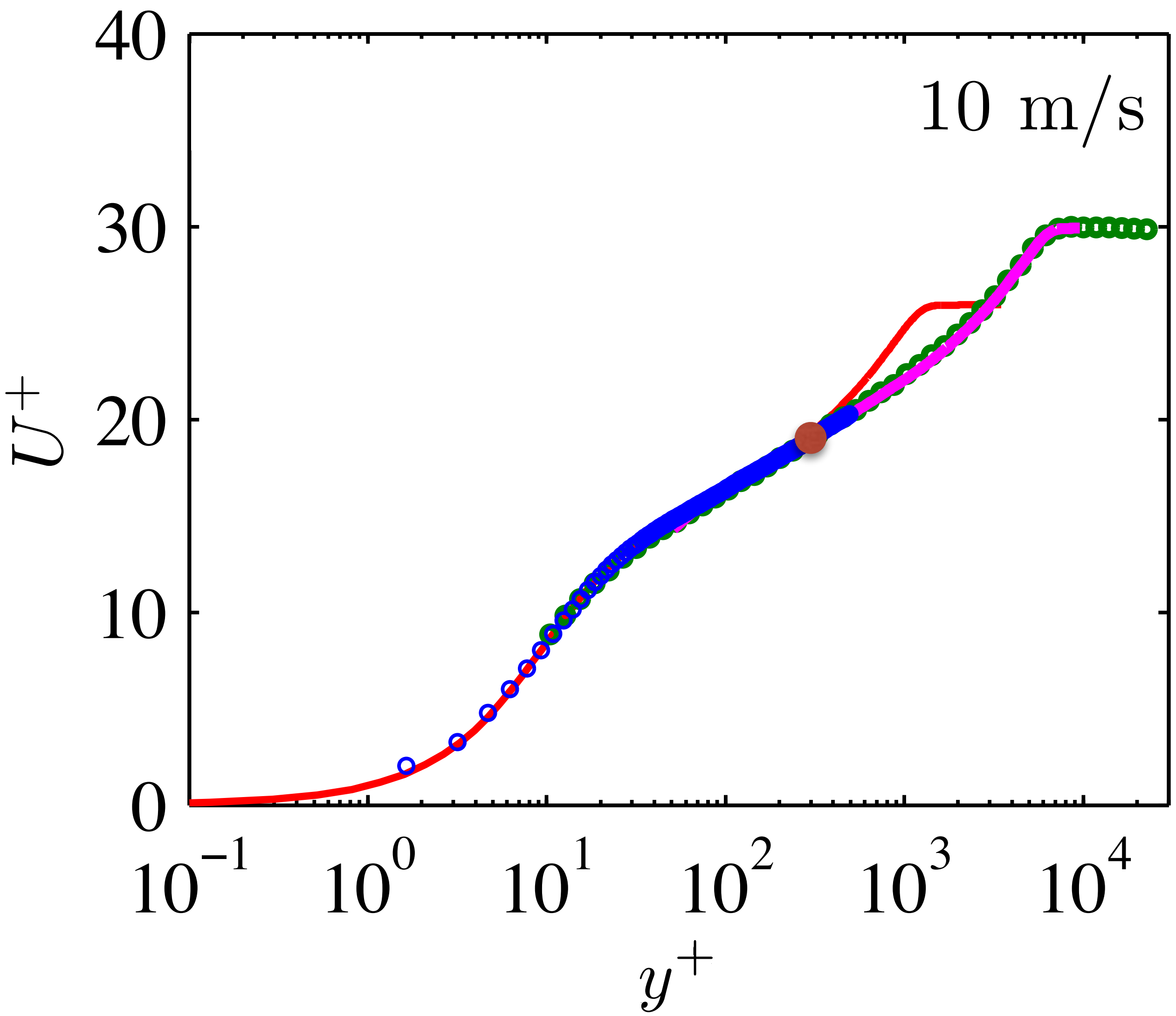
$y^+ = 300$



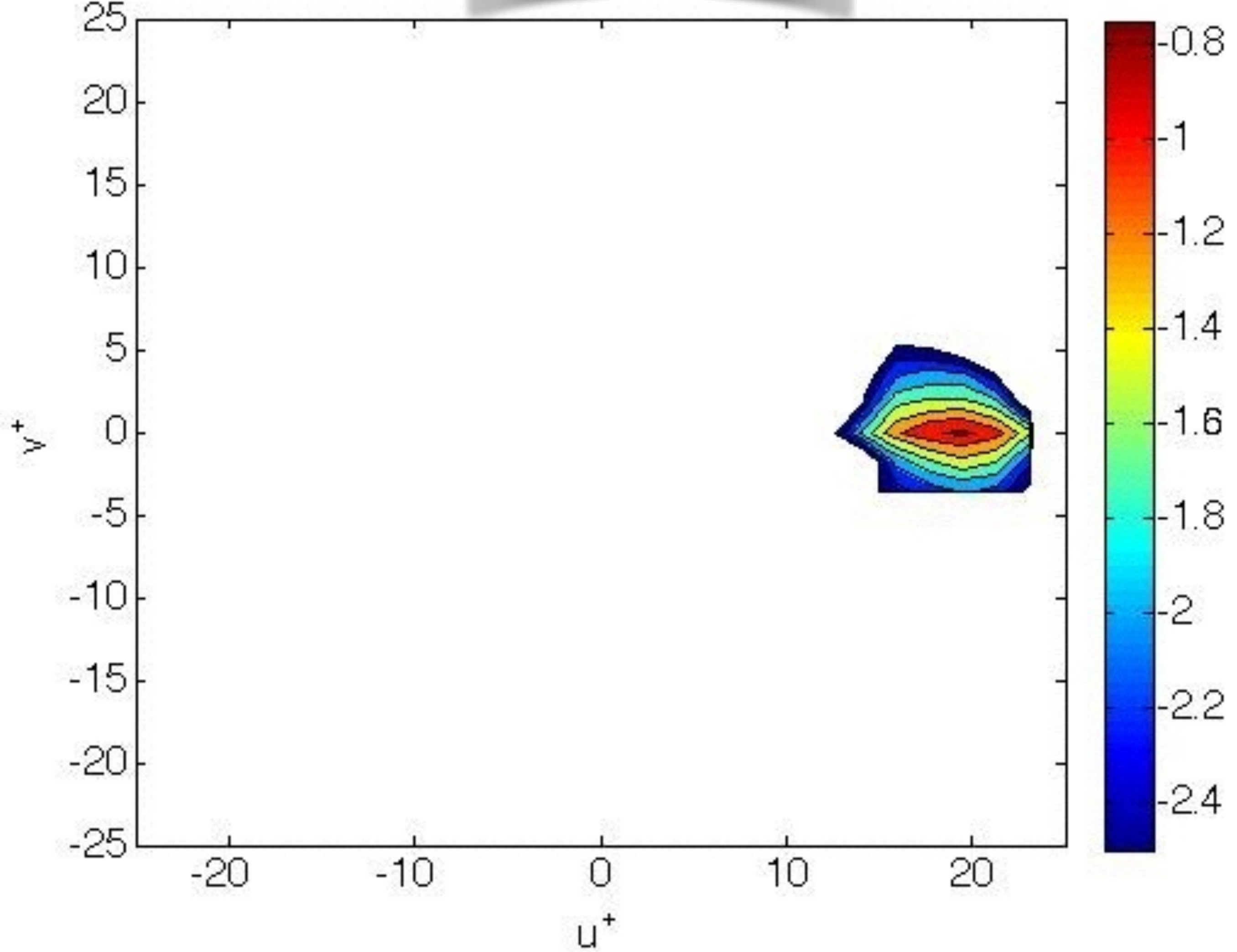
Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at $Re_\tau = 8000$

- 2C-2D JPDF:

$$IV = l_x^+ \times l_y^+ = 9.5 \times 1.2$$



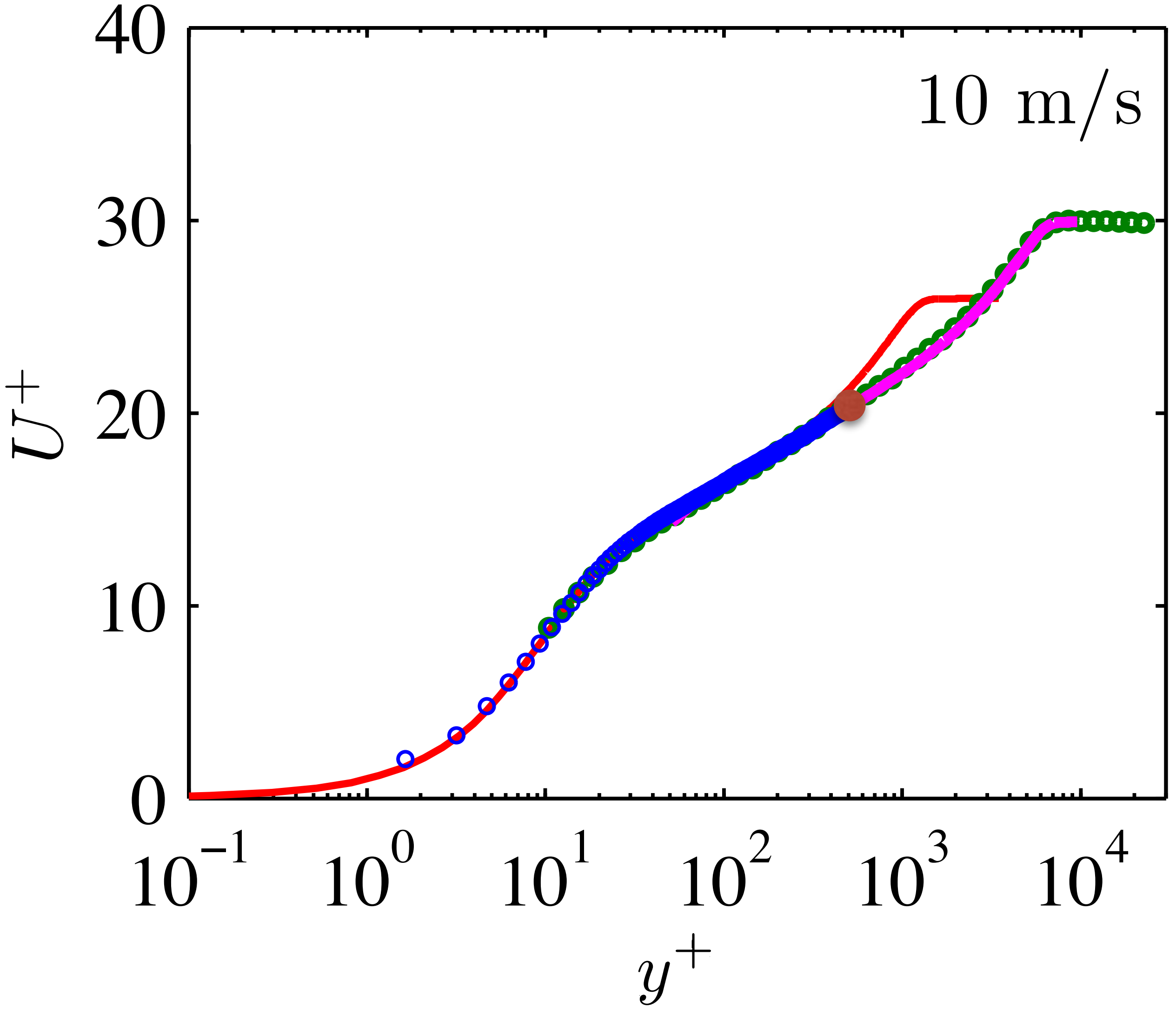
$y^+ = 400$



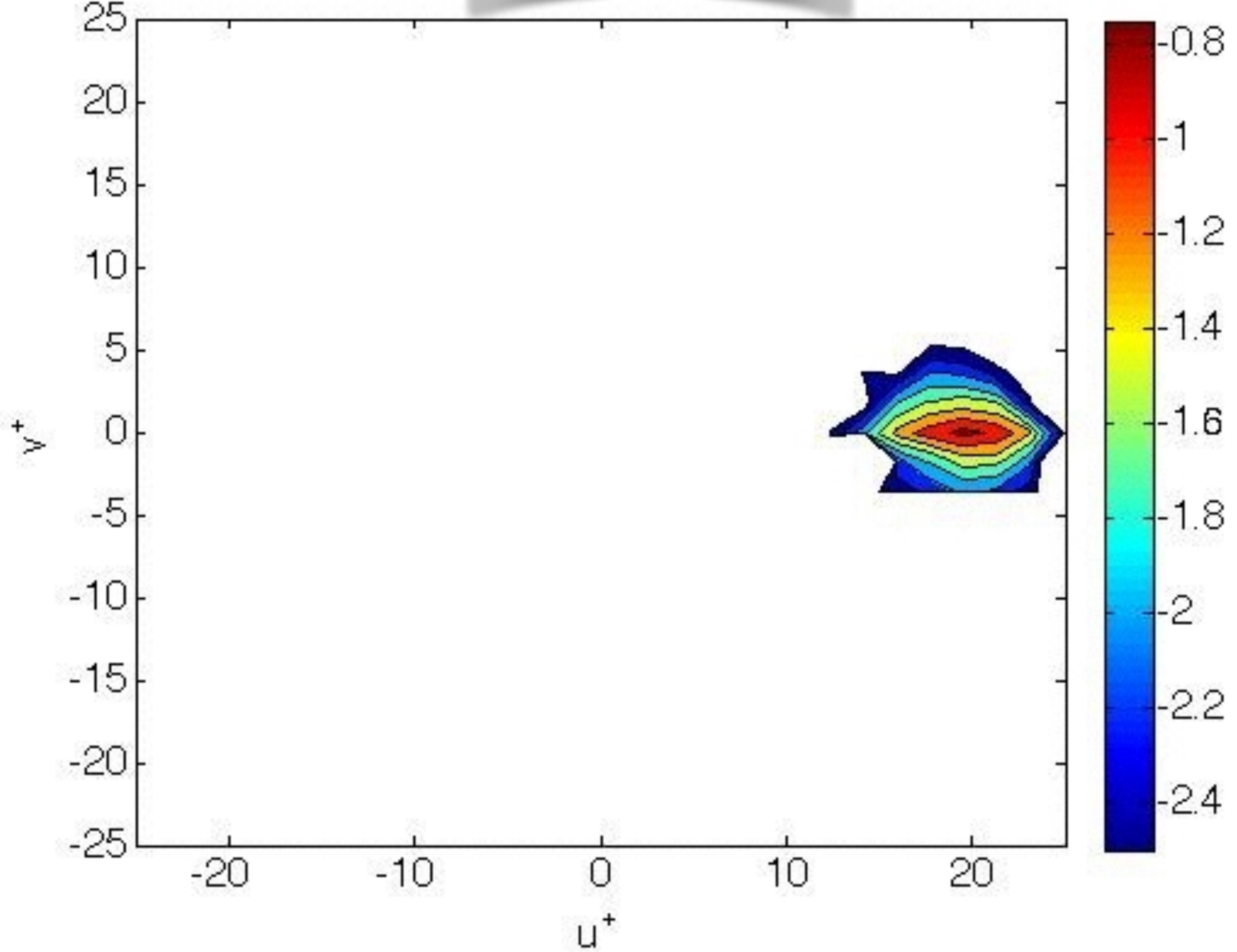
Application: 2C-2D JPDF measurements of High Reynolds number ZPG TBL at Re_τ = 8000

- 2C-2D JPDF:

$$IV = l_x^+ \times l_y^+ = 9.5 \times 1.2$$



$y^+ = 500$



Remarks …

- the velocity JPDF is directly related via a simple relationship to the expected single-particle cross-correlation which takes into account finite particle size
- for small particles, $d < 2\ px$, the approximation that the JPDF is equal to the expected single-particle cross-correlation yields accurate results up to second moments
- technique has been applied to ZPG TBL at $Re_\tau = 8{,}000$ to measure 2C-2D JPDF
- velocity dynamic range is a problem?
  ⇒ need to increase spatial dynamic range for a given fixed *IV?*