Simple Models for Wall Turbulence

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What do we want from a model?

Navier-Stokes equations

some ideal model

scalings, symmetries

structure

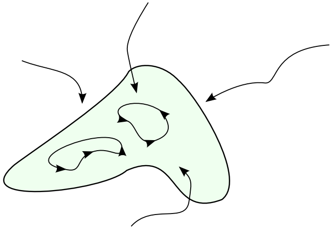
statistical moments

drag

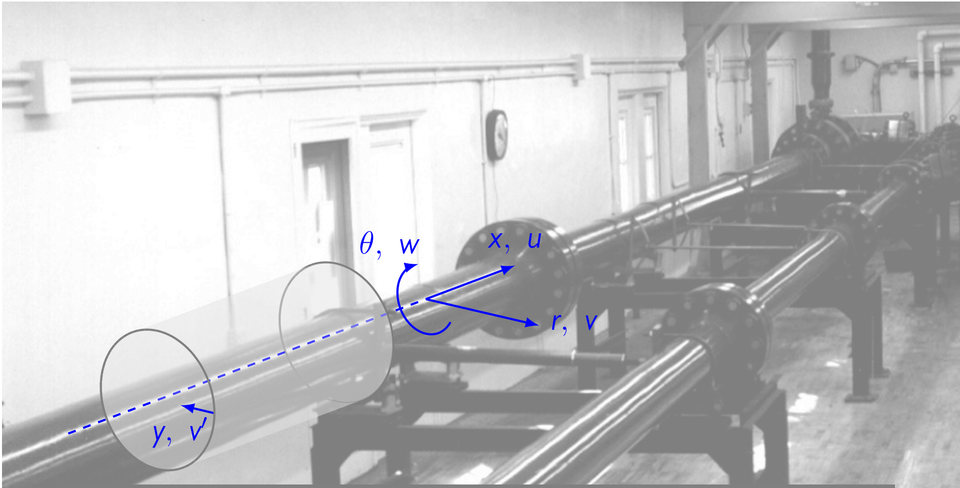
energy spectra

$$\begin{tikzcd}
\text{Coherent Structure} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \text{Important dynamics} \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
\text{Low-order model} \arrow[r, leftrightarrow] & \text{Efficient basis}
\end{tikzcd}$$

What to throw away?



- In time, trajectories approach some turbulent attractor that defines dynamics
- this should be the statistically steady state
- we wish to gain insight about long-time behaviour



Fourier decomposition into radially-varying travelling waves

$$\mathbf{u}(r; x, \theta, t) = \sum_{k,n,\omega} \mathbf{u}(r; k, n, \omega) e^{i(kx+\theta n-\omega t)}$$

mean: McKeon & Morrison 2004

NSE in Fourier domain

Fourier modes for velocity in all three homogeneous directions

$$\mathbf{u}(r; x, \theta, t) = \sum_{k,n,\omega} \mathbf{u}_{\mathbf{k}}(r) e^{i(kx+\theta n-\omega t)}$$

$$\mathbf{u}_{\mathbf{k}}(r) := \hat{\mathbf{u}}(r; k, n, \omega)$$

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$$\mathbf{u}_\mathbf{k}(r) := \hat{\mathbf{u}}(r; k, n, \omega)$$

subtract out steady-state / $\mathbf{u}_0$ / mean / $(k, n, \omega) = (0, 0, 0)$
and substitute for nonlinear term

$$\mathbf{f}(r; x, \theta, t) := -\left(\mathbf{u} - \mathbf{u}_0(r)\right) \cdot \nabla \left(\mathbf{u} - \mathbf{u}_0(r)\right)$$

$$\mathbf{f}(r; x, \theta, t) := \sum_{k,n,\omega} \mathbf{f}_\mathbf{k}(r) e^{i(kx+\theta n-\omega t)}$$

Steady-state equation

The equation for $(k, n, \omega) = (0, 0, 0)$ gives the mean

Re stress gradients supporting mean

$$0 = \boxed{\mathbf{f}_0(r)} - \mathbf{u}_0(r) \cdot \nabla \mathbf{u}_0(r) + Re^{-1} \nabla^2 \boxed{\mathbf{u}_0(r)}$$

time-space ave velocity

Clear interpretation for linear operators formed about the mean profile – *this is **not** small perturbation analysis around a fixed point*

Equation for fluctuations at (k, n, ω)

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$$
\begin{aligned}
-i\omega\mathbf{u}_{\mathbf{k}}(r) = &- \mathbf{u}_0(r)\cdot\nabla\mathbf{u}_{\mathbf{k}}(r) \\
&- \mathbf{u}_{\mathbf{k}}(r)\cdot\nabla\mathbf{u}_0(r) \\
&+ Re^{-1}\nabla^2\mathbf{u}_{\mathbf{k}}(r) - \nabla p_{\mathbf{k}}(r) \\
&+ \mathbf{f}_{\mathbf{k}}(r) \\
\\
= &\ \mathrm{L}_{\mathbf{k}}(\mathbf{u}_0)\mathbf{u}_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}}(r)
\end{aligned}
$$

Equation for fluctuations at (k, n, ω)

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interaction between scales

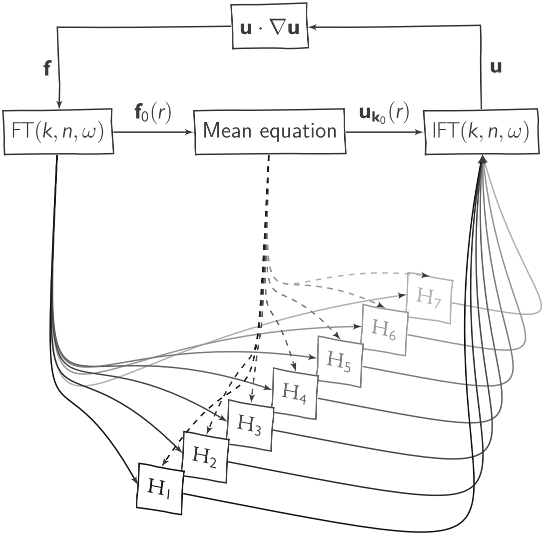
$$\mathbf{u}_{\mathbf{k}}(r) = \boxed{\mathrm{H}_{\mathbf{k}}}\ \boxed{\mathbf{f}_{\mathbf{k}}(r)}$$

linear resolvent

The resolvent $\mathrm{H}_{\mathbf{k}} = (i\omega - \mathrm{L}_{\mathbf{k}})^{-1}$ is the *frequency-domain* (so travelling waves) transfer function from nonlinear interaction between scales to velocity field

$\mathbf{u}_{\mathbf{k}}$ requires $\mathbf{u}_0$ requires $\mathbf{u}_{\mathbf{k}}$... close by assuming $\mathbf{u}_0$

NSE as a network of resolvents



$$\mathbf{f}_{\mathbf{k}}(r) \longrightarrow \boxed{\mathrm{H}_{\mathbf{k}}} \longrightarrow \mathbf{u}_{\mathbf{k}}(r)$$

Resolvent

$$\mathrm{H}_{\mathbf{k}} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

$$D = \partial_r^2 + r^{-1}\partial_r - r^{-2}(n^2 + 1) - k^2, \text{ states } (u_r, u_\theta, u_x)$$

Meseguer & Trefethen JCP 2003

$\mathbf{f_k}(r) \longrightarrow \boxed{\mathrm{H}_\mathbf{k}} \longrightarrow \mathbf{u_k}(r)$

Non-normality

$$\mathrm{H}_\mathbf{k} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

- Mean shear term is main source of non-normality
- Only net source of fluctuation energy
- Translational symmetries $\Leftrightarrow$ normal operator

McKeon & Sharma JFM 2010

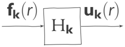
$\mathbf{f_k}(r) \longrightarrow \boxed{\mathrm{H}_\mathbf{k}} \longrightarrow \mathbf{u_k}(r)$

Critical layer response

$$\mathrm{H}_\mathbf{k} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

- as $c = \omega/k \to \mathbf{u}_0$ and $Re \to \infty$, $\qquad \|\mathrm{H}_\mathbf{k}\| \to \infty$
- Energetic response at critical layer becomes very important at high Re.
- decouples from shear mechanism
- response becomes more localised around critical layer

McKeon & Sharma JFM 2010



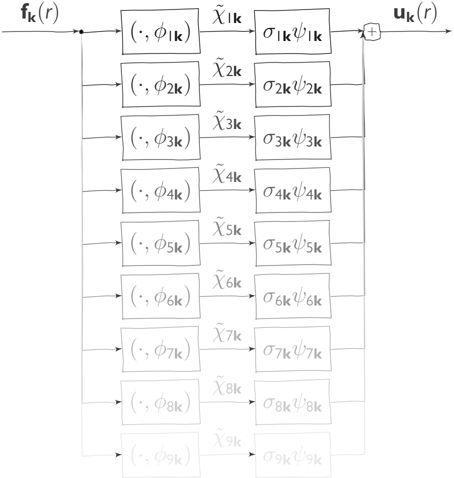
Approximating a single resolvent

SVD approximates an operator by directions of principal gain

$$\mathrm{H}_{\mathbf{k}} = \sum_{m=1}^{\infty} \psi_{m\mathbf{k}}(r)\sigma_{m\mathbf{k}}\phi^*_{m\mathbf{k}}(r)$$

$$\delta_{l,m} = (\phi^*_{l\mathbf{k}}(r), \phi^*_{m\mathbf{k}}(r))_r$$
$$\delta_{l,m} = (\psi^*_{l\mathbf{k}}(r), \psi^*_{m\mathbf{k}}(r))_r$$
$$\sigma_m \geq \sigma_{m+1} \geq 0$$

Each σ_m is a (real) gain,
σ_1 is the maximum gain.



$\mathbf{f_k}(r) \longrightarrow \boxed{\mathrm{H}_\mathbf{k}} \longrightarrow \mathbf{u_k}(r)$

Approximation of $\mathrm{H}_\mathbf{k}$ (by gain)

- This gives *radial form (or structure)* of velocity field at **k** — natural radial basis
- Reduces NSE solution to a weighted sum of response modes

$$\mathbf{u}(x, \theta, r, t) \approx \sum_{\mathbf{k}} \chi_{\mathrm{l}\mathbf{k}} \psi_{\mathrm{l}\mathbf{k}}(r) \mathrm{e}^{i(kx+n\theta-\omega t)}$$

Ways to proceed

1. assume unit forcing for all $\mathbf{k}$ — spectra
2. analysis of resolvent for symmetries — scaling
3. manually pick combinations of $\chi_{\mathbf{k}}$ — structure
4. solve $\chi_j = \sum_{ab} \sigma_j N_{jab} \chi_a \chi_b$ — nonlinear solutions

Response mode location, $(k, n) = (1, 10)$

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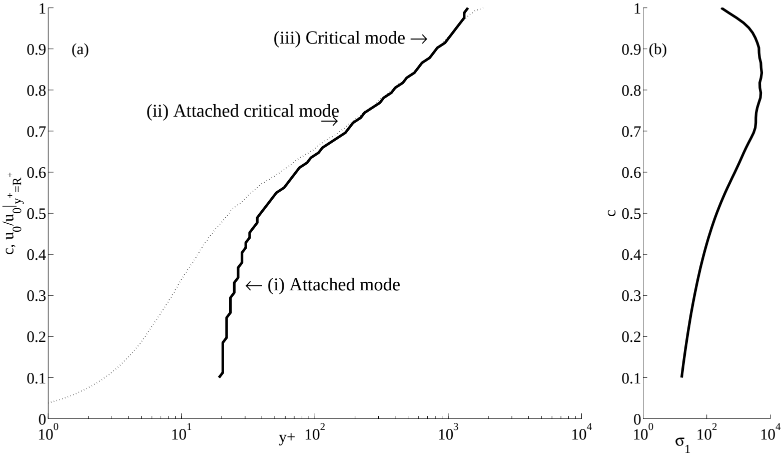
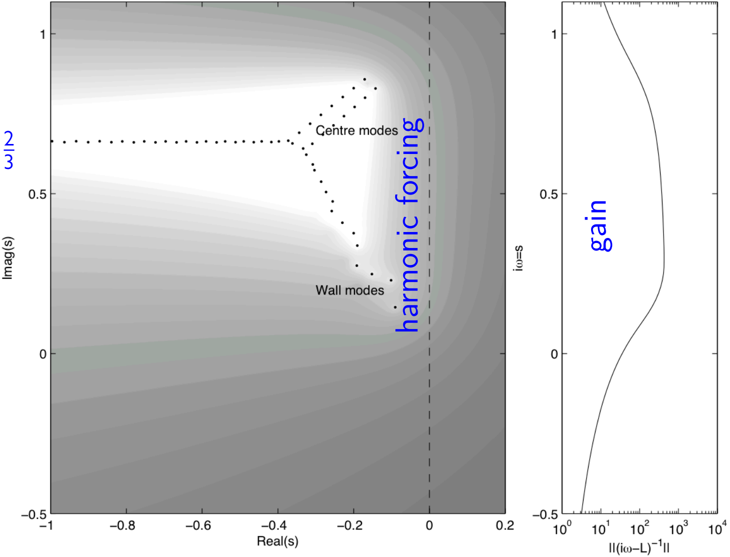


McKeon & Sharma JFM 2010



$\frac{2}{3}$
1
0.5
0
−0.5
Imag(s)
Centre modes
Wall modes
harmonic forcing
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
Real(s)
iω=s
gain
10^0
10^1
10^2
10^3
10^4
$\|(i\omega - L)^{-1}\|$

Spectra: direct from resolvent gains

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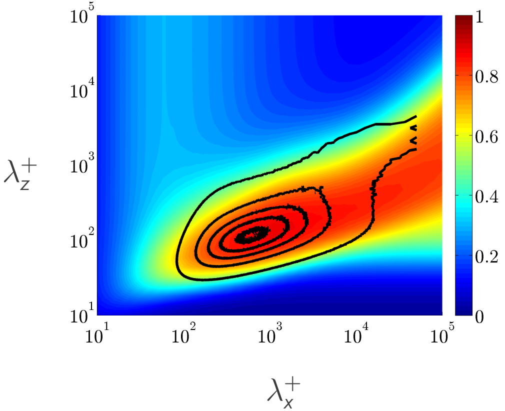
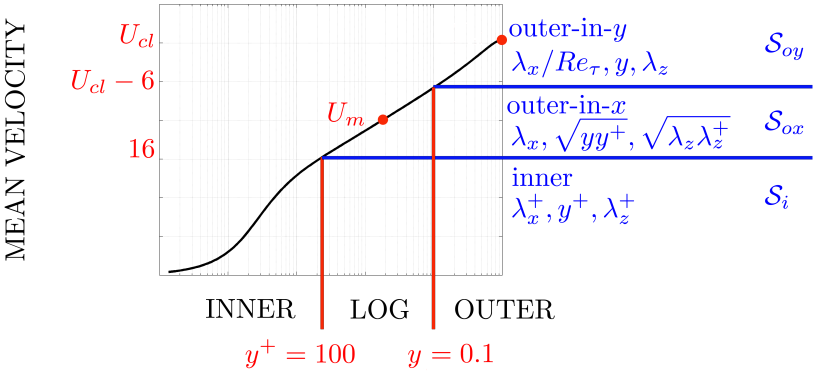
E_{uu} for unit white noise forcing on first two modes only at $y^+ = 15$ vs DNS



(in a channel) Moarref & al JFM 2013

DNS (lines): Hoyas & Jimenez PoF 2006

Scaling: symmetries in the resolvent



- mean profile scaling regions induce symmetries in resolvent
- reveals scalings of response modes

Moarref & al JFM 2013

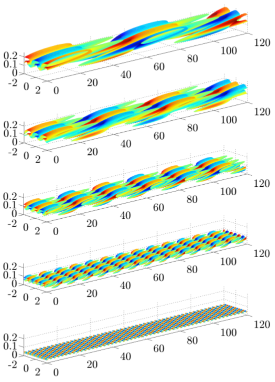
Scaling: geometric self-similarity

Under assumptions:

1. modes local in y
2. critical layer term scales geometrically $U(y) - c = g(y/y_c)$
3. $ik_x(U(y) - c)$ term balances with $Re_\tau^{-1}\Delta$ term

log region is necessary to obtain invariant resolvent.

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{bmatrix} y_c^+ y_c \overline{H}_{11} & (y_c^+)^2 y_c \overline{H}_{12} & (y_c^+)^2 y_c \overline{H}_{13} \\ y_c \overline{H}_{21} & y_c^+ y_c \overline{H}_{22} & y_c^+ y_c \overline{H}_{23} \\ y_c \overline{H}_{31} & y_c^+ y_c \overline{H}_{32} & y_c^+ y_c \overline{H}_{33} \end{bmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$



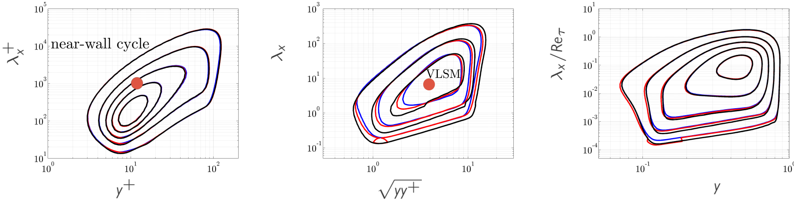
Moarref & al JFM 2013

Similar arguments derive classical inner and outer Re scalings for resolvent modes

Scaling of u, λ_x streamwise energy spectra

$$E_{uu}(y;k_x) = \overbrace{\boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\mathrm{d}c}}^{\text{inner}} + \overbrace{\boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\mathrm{d}c}}^{\text{self-similar (analytical)}} + \overbrace{\boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\mathrm{d}c}}^{\text{outer}}$$

E_{uu}/Re_τ^2



$Re_\tau = 3333,\ 10000,\ 30000$

Scaling of u, λ_x streamwise energy spectra

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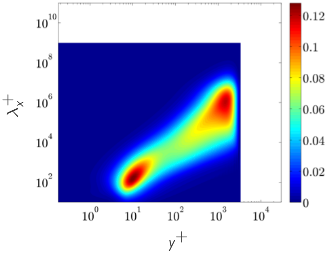
E_{uu}/Re_τ^2

$Re_\tau = 3333$

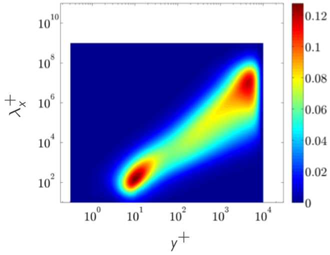


Scaling of u, λ_x streamwise energy spectra

$$E_{uu}(y;k_x) = \overset{\text{inner}}{\boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\mathrm{d}c}} + \overset{\text{self-similar (analytical)}}{\boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\mathrm{d}c}} + \overset{\text{outer}}{\boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\mathrm{d}c}}$$

E_{uu}/Re_τ^2

$Re_\tau = 10000$



Scaling of u, λ_x streamwise energy spectra

inner

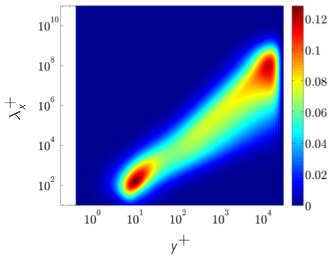
self-similar (analytical)

outer

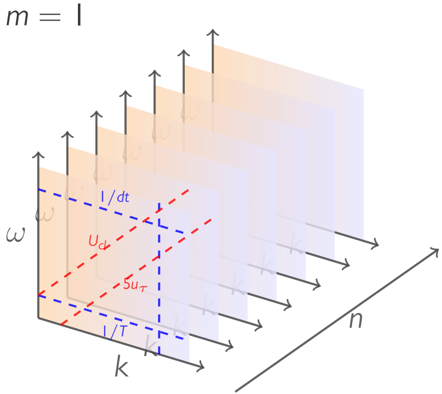
$$E_{uu}(y;k_x) = \boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\mathrm{d}c} + \boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\mathrm{d}c} + \boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\mathrm{d}c}$$

E_{uu}/Re_τ^2

$Re_\tau = 30000$

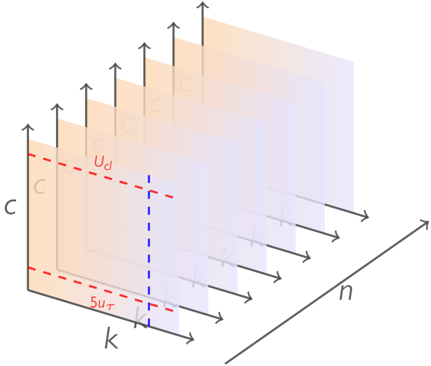


Turbulence as sheets of coefficients

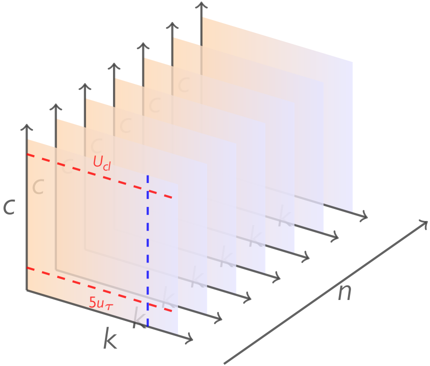


Turbulence as sheets of coefficients

$m = 1$



$m = 2$



Hairpin packet analogy to near-wall cycle

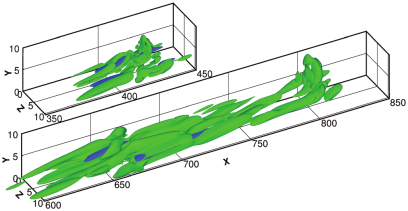
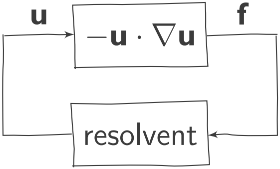


FIG. 2. (Color online) Snapshots of the streamwise component of the perturbation (darker surfaces, blue online, for u ¼À0:13) and of the Q-criterion (lighter surfaces, green online) at t ¼ 300 and t ¼ 700 (top and bottom, respectively) obtained by the DNS initialized with the nonlinear optimal perturbation with E_0 ¼ 0:004444275.

Cherubini & al PoF 2011



- Recently found 'edge states' look like hairpin packets
- Triadically compatible mode combinations can self-sustain via nonlinearity

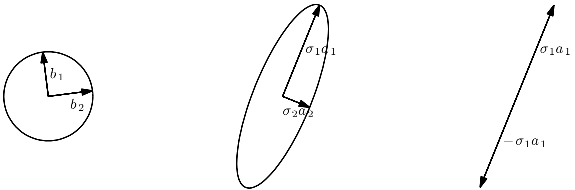
Conclusions

- localisation of modes around critical layer associated with Taylor's hypothesis
- symmetries in resolvent induced by assumed mean results in scaling behaviour of resolvent modes and spectra
- attached-critical behaviour associated with impingement of S-branch on imag axis
- non-orthogonality of ϕ_k to $\mathbf{f_k}$ fixes coefficients; permits self-sustaining mode combinations

Extra stuff

The singular value decomposition

SVD gives L_2 gain-optimal low-dimensional approximation of an operator.



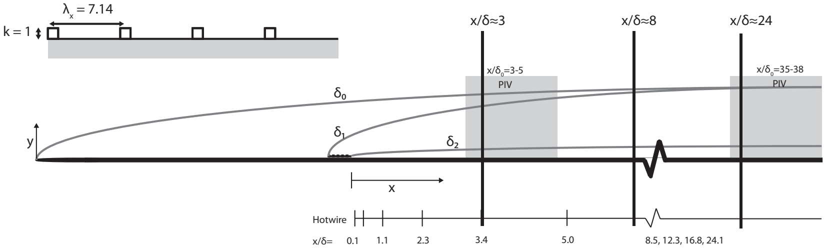
$$M = \begin{bmatrix} 0.5 & 0.9 \\ -0.5 & 2 \end{bmatrix} = ASB^*,$$

$$A = \begin{bmatrix} 0.3761 & 0.9266 \\ 0.9266 & -0.3761 \end{bmatrix}, \quad S = \begin{bmatrix} 2.2089 & 0 \\ 0 & 0.6564 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1246 & 0.9922 \\ 0.9922 & 0.1246 \end{bmatrix}.$$

Optimal rank-1 approximation:

$$\tilde{M} = A \begin{bmatrix} 2.2089 & 0 \\ 0 & 0 \end{bmatrix} B^*$$

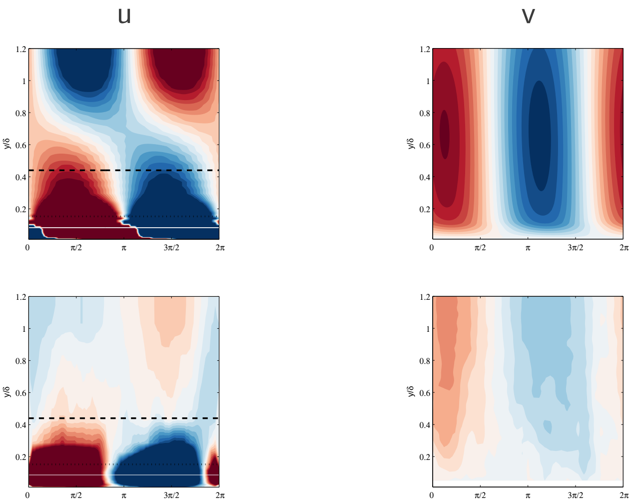
Exciting a large-scale motion



Jacobi & McKeon JFM 2011

McKeon & al PoF 2013

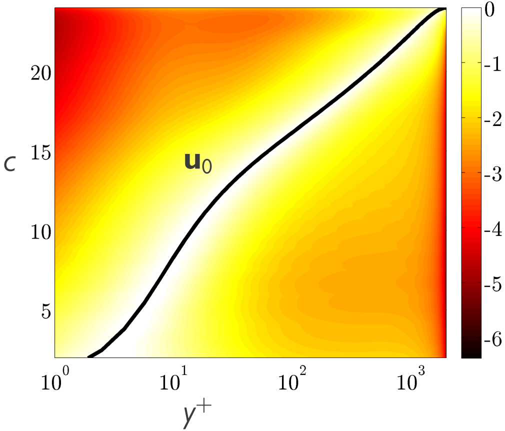
Exciting a large-scale motion



$3\delta_0$ downstream, predicted and measured

McKeon & al PoF 2013

Localisation of response at critical layer



$\log(Re_\tau^{-2}E_{uu}(y,c))$; normalised; $Re_\tau = 2003$

Moarref et al JFM 2013

The nonlinear term

$$\mathbf{f}_{ab} := -\mathbf{u}_a \cdot \nabla \mathbf{u}_b$$

The full quadratic spectral transfer function for $\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_c + \ldots$ can be built up from this as the sum

$$\mathbf{f} = \sum_{i=\{a,b,c,\ldots\}} \sum_{j=\{a,b,c,\ldots\}} \mathbf{f}_{ij} = \begin{array}{cccc} \mathbf{f}_{aa} & +\mathbf{f}_{ab} & +\mathbf{f}_{ac} & \ldots \\ \hline +\mathbf{f}_{ba} & +\mathbf{f}_{bb} & +\mathbf{f}_{bc} & \\ \hline +\mathbf{f}_{ca} & +\mathbf{f}_{cb} & +\mathbf{f}_{cc} & \\ \hline \vdots & & & \ddots \end{array}$$

Mode interactions (in $x,\ \theta,\ t$)

Wave combinations provide forcing at the sum of wavenumbers

$$\mathbf{u}_a + \mathbf{u}_b \to \mathbf{f}_{a+b}$$

	k	n	$c = \omega/k$
$\mathbf{k}_\text{I}$	6	6	$2/3$
$-\mathbf{k}_\text{I}$	-6	-6	$2/3$
$\mathbf{k}_{-\text{I}+\text{I}}$	0	0	$\omega = 0$

Any real mode interacts with the mean

Mode interactions (in x, θ, t)

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Wave combinations provide forcing at the sum of wavenumbers

$$\mathbf{u}_a + \mathbf{u}_b \rightarrow \mathbf{f}_{a+b}$$

	k	n	$c = \omega/k$
$\mathbf{k}_l$	6	6	$2/3$
$-\mathbf{k}_l$	-6	-6	$2/3$
$\mathbf{k}_{-l+l}$	0	0	$\omega = 0$

Any real mode interacts with the mean

	k	n	c
$\mathbf{k}_l$	6	6	$2/3$
$\mathbf{k}_2$	1	6	$2/3$
$\mathbf{k}_{l+2}$	7	12	$2/3$

The idealised hairpin packet can self-interact
Just requires that $(\mathbf{f}_a, \phi_a)_r \neq 0$

Mode interaction (in r)

In wall-normal direction, forcing mode $\phi_{\mathbf{k}}(r)$ is not typically orthogonal to $\mathbf{f}_{\mathbf{k}}(r)$

Modes at triadically consistent $\mathbf{k}$ can self-support with large enough amplitude.

Sharma & McKeon AIAA 2013

Mode interaction (wall-normal cascade)

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In wall-normal direction, $(\psi_l(r), \phi_m(r))_r \neq \delta_{lm}$...

Production due to a mode combination is (in matrix form)

$$\frac{dE}{dt} = (\mathbf{f}, \mathbf{u}) = \chi^* \Phi^* \Psi \Sigma \chi \approx \sum_m |\chi_m|^2 (\phi_m, \psi_m) \sigma_{1,m}$$

- dominated by leading mode (factor of σ_1)
- response and forcing modes not orthogonal $\Rightarrow$
- transfer of momentum across r and between modes