Modeling and Simulation in Rotationally Constrained (Convective) Flows

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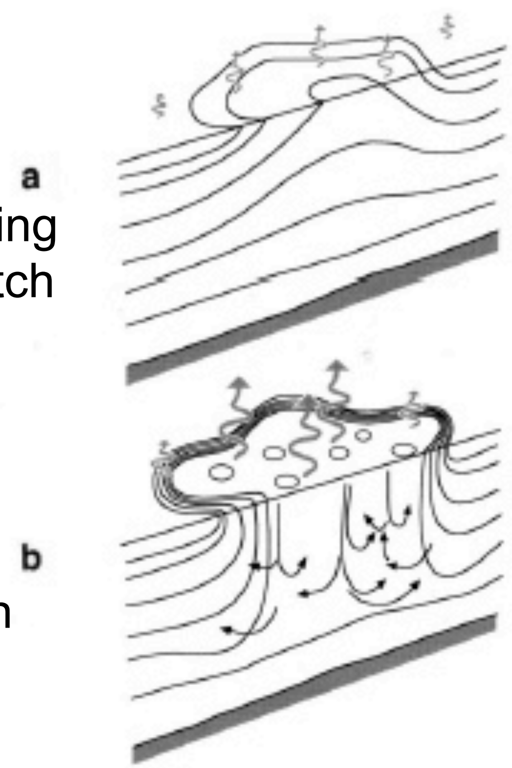
NSF EAR CSEDI

Outline

- Motivation
- Slow Manifold Equations: Nonhydrostatic QG limit
- Application - Simulations

Rotationally Constrained Convective Flows in GAFD $Ro \ll 1$

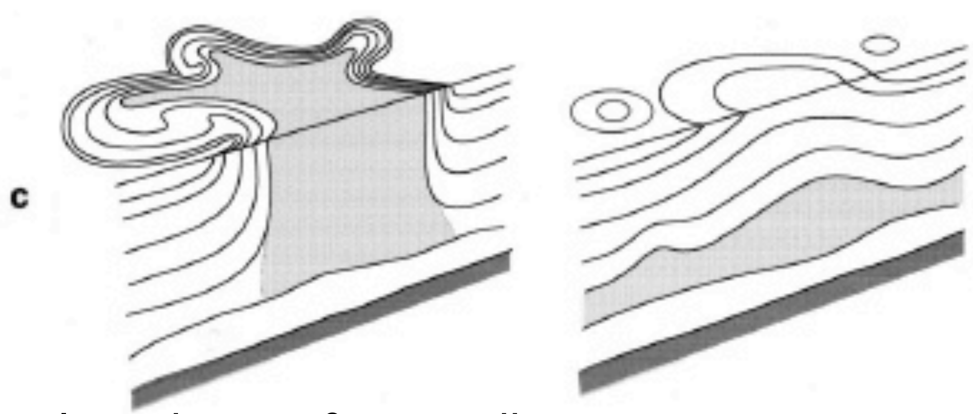
Marshall and Schott: OPEN-OCEAN CONVECTION • 5



Preconditioning
~100 Km patch

$Ro \sim 0.1 - 0.4$
$U \sim 0.05\ m/s$
$\Omega \sim 7 \times 10^{-5} rad/s$
$L \sim 2km$

Deep
Convection



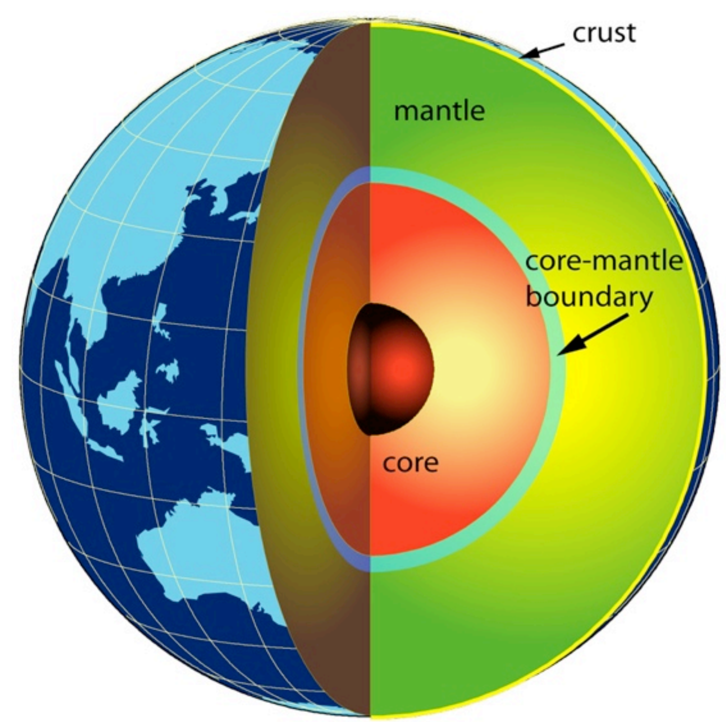
Lateral exchange & spreading

$$Ro = \frac{t_\Omega}{t_{adv}} = \frac{U}{2\Omega L}$$

large-scale flow generation on Giant Planets

$Ro \sim 10^{-2}$
$U \sim 100m/s$
$\Omega \sim 2 \times 10^{-4}\ rad/s$
$L \sim 15\ Mm$

turbulence primary driver for geomagnetic field



$Ro \sim 10^{-7}$
$U \sim 3 \times 10^{-4}\ m/s$
$\Omega \sim 7 \times 10^{-5}\ rad/s$
$L \sim 2260\ km$

Rotationally Constrained Convective Flows in GAFD $Ro \ll 1$

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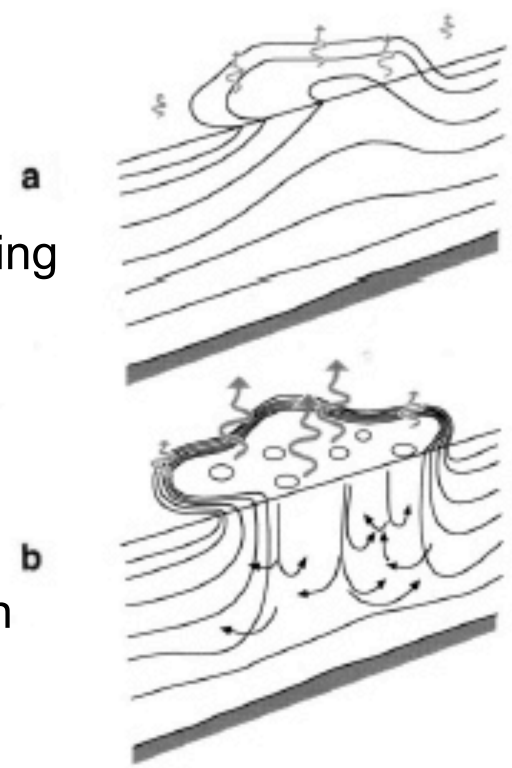


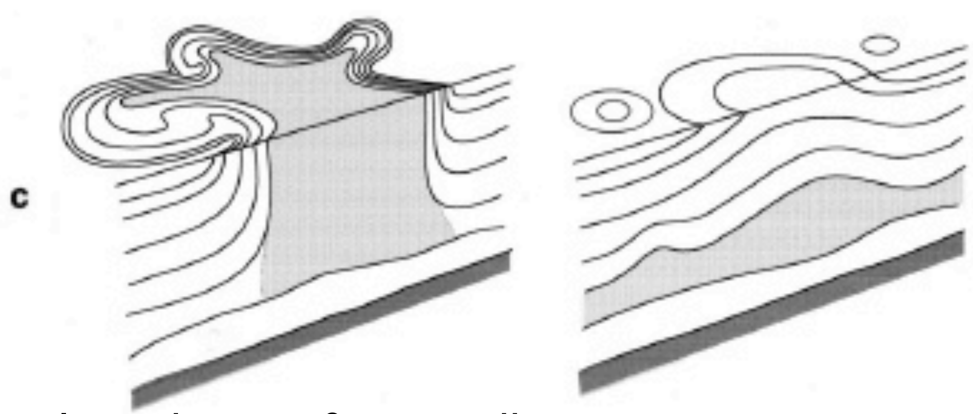
Preconditioning

Deep Convection

$Ro \sim 0.1 - 0.4$

$Re \sim 10^8$

$Ek \sim 10^{-9}$



Lateral exchange & spreading

$$Ro = \frac{t_\Omega}{t_{adv}} = \frac{U}{2\Omega L}$$

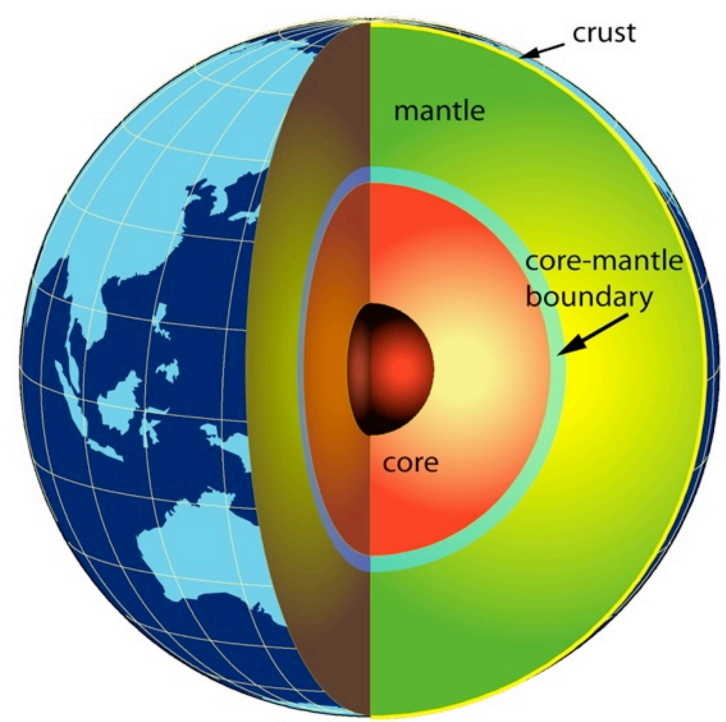
large-scale flow generation on Giant Planets

$Ro \sim 10^{-2}$

$Re \sim 10^{16}$

$Ek \sim 10^{-18}$

turbulence primary driver for geomagnetic field



$Ro \sim 10^{-7}$

$Re \sim 10^8$

$Ek \sim 10^{-15}$

Nonhydrostatic Investigations: Canonical Configurations

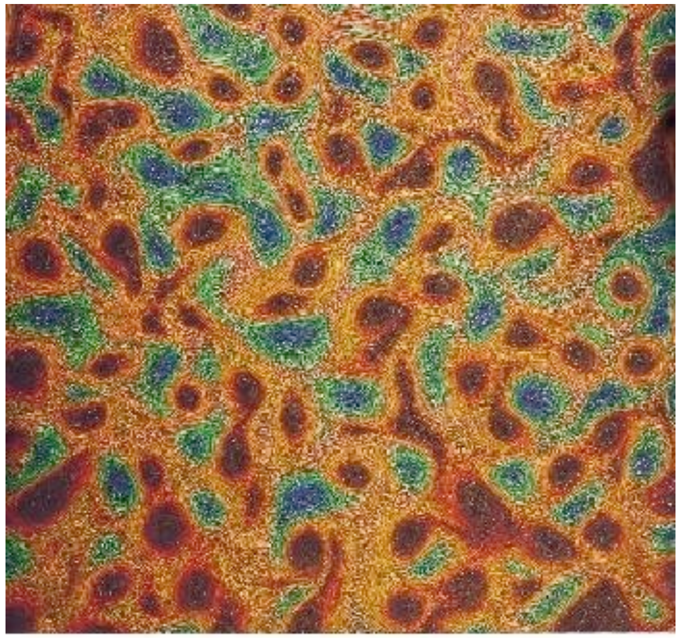
- Non-hydrostatic, rotationally constrained flows characterized by columnar structures
- probing low Ro, high Re challenging
  - experimentation: restricted by mechanical and fluid properties
  - DNS: restricted by spatiotemporal resolution constraints.

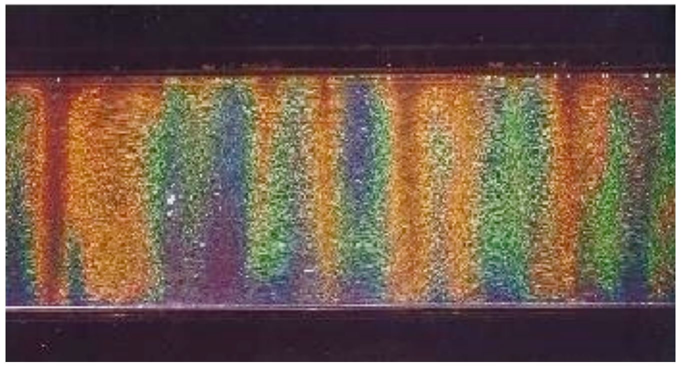
Maxworthy & Narimosa JPO 1994

nonhomog. heat source

Liu & Ecke PRE '09; King et al Nature '09

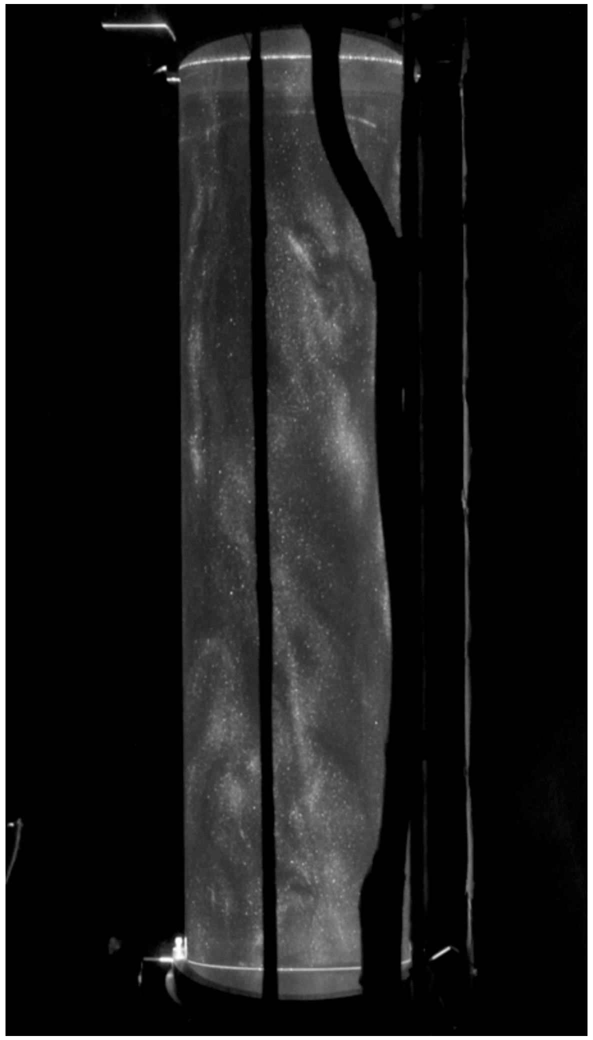
Sakai, JFM 1997: $RaE^{4/3}$ = 36, Ro≈0.1, σ = 7



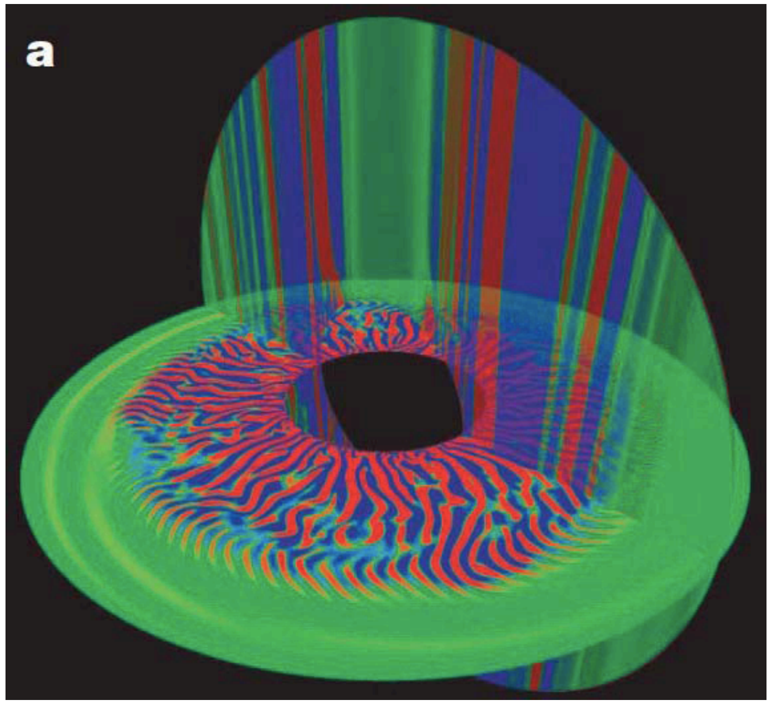


Taylor columns

Aurnou, $RaE^{4/3}$ = 755, Ro≈0.13, σ = 7



Plumes



Axial vorticity, E~2e-7 (Kageyama et al Nature 2008)
Ra=1.5e10

Navier-Stokes Equations (Non-dimensional Characterization)

- Generic non-dimensionalization: $L, U, \Delta T, P$

$$D_t \boldsymbol{u} + Ro^{-1} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla p + \Gamma b \widehat{\boldsymbol{z}} + Re^{-1} \nabla^2 \boldsymbol{u} + \boldsymbol{S}$$

$$D_t b - \Gamma^{-1} Fr^{-2} w \partial_z \rho(z) = Pe^{-1} \nabla^2 b$$

$$\nabla \cdot \boldsymbol{u} = 0$$

where $D_t := \partial_t + \boldsymbol{u} \cdot \nabla$ with $(\boldsymbol{u}, p, b)$ for velocity, pressure & buoyancy fields.

- Non-dimensional Parameters:

Rossby Number	$Ro = \frac{U}{2\Omega L}$	Ekman Number	$Ek = \frac{Ro}{Re} = \frac{\nu}{2\Omega L^2}$
Euler Number	$Eu = \frac{P}{\rho_0 U^2}$	Reynolds Number	$Re = \frac{UL}{\nu}$
Buoyancy Number	$\Gamma = \frac{g\alpha \Delta T L}{U^2}$	Péclet Number	$Pe = \frac{UL}{\kappa}$
Froude Number	$Fr = \frac{U}{NL}$		

Navier-Stokes Equations (Incompressible Fluid)

- Generic non-dimensionalization: $L, U, \Delta T, P$

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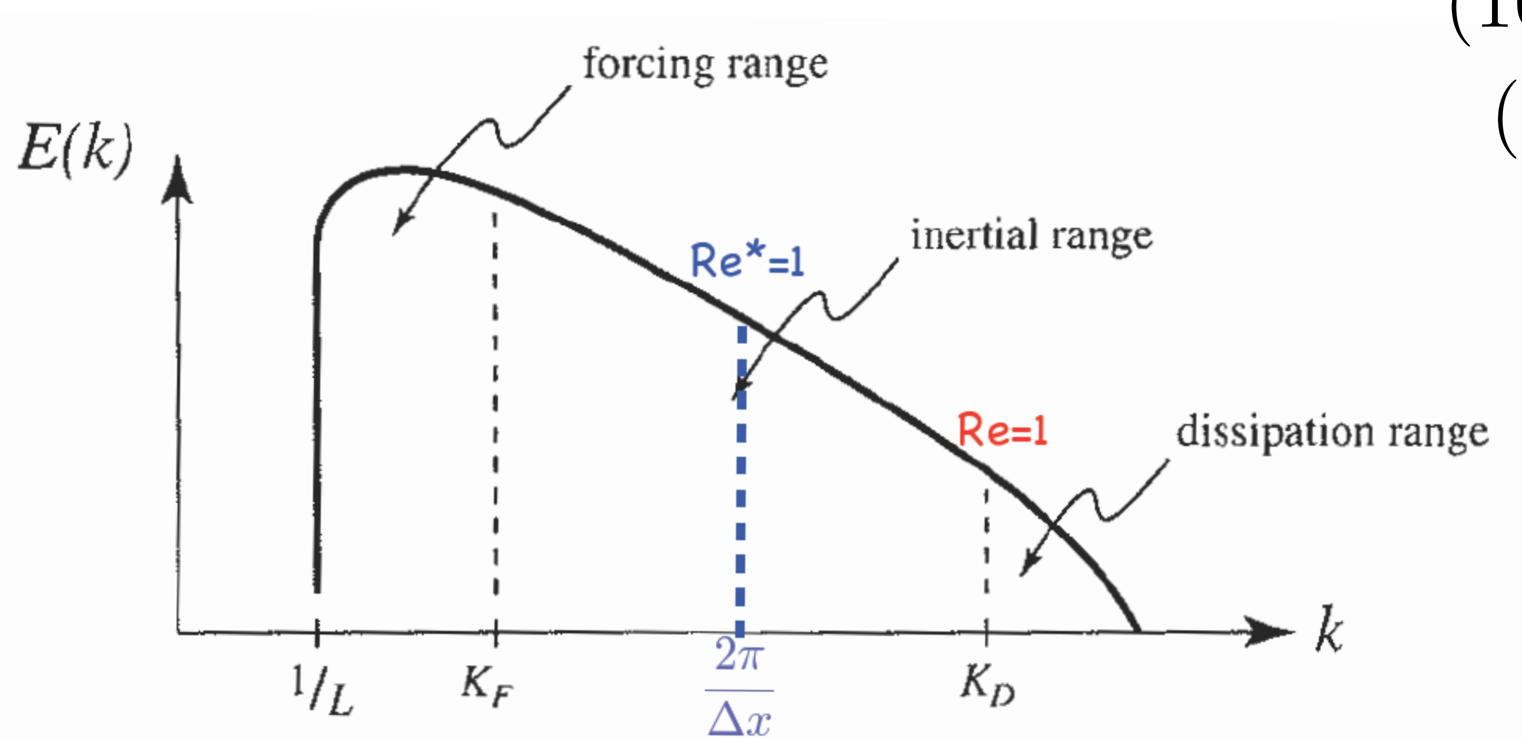
- Turbulence Challenge: d.o.f. (grid-pts/modes) $\Rightarrow N^3 \sim Re^{\frac{9}{4}}$ (Pope, 2000)

$(10^{6+})^3 \sim (10^{8+})^{\frac{9}{4}} \Rightarrow$ GAFD

$(10^3)^3 \sim (10^4)^{\frac{9}{4}} \Rightarrow$ DNS



$\mathcal{T} \sim 2Re^3 / P_{flop\ rate}$

$30d \sim 2(10^8)^3 / 10^{23} \Rightarrow$ GAFD

Moore's Law $\Rightarrow$ 70 yrs away

Navier-Stokes Equations (Incompressible Fluid)

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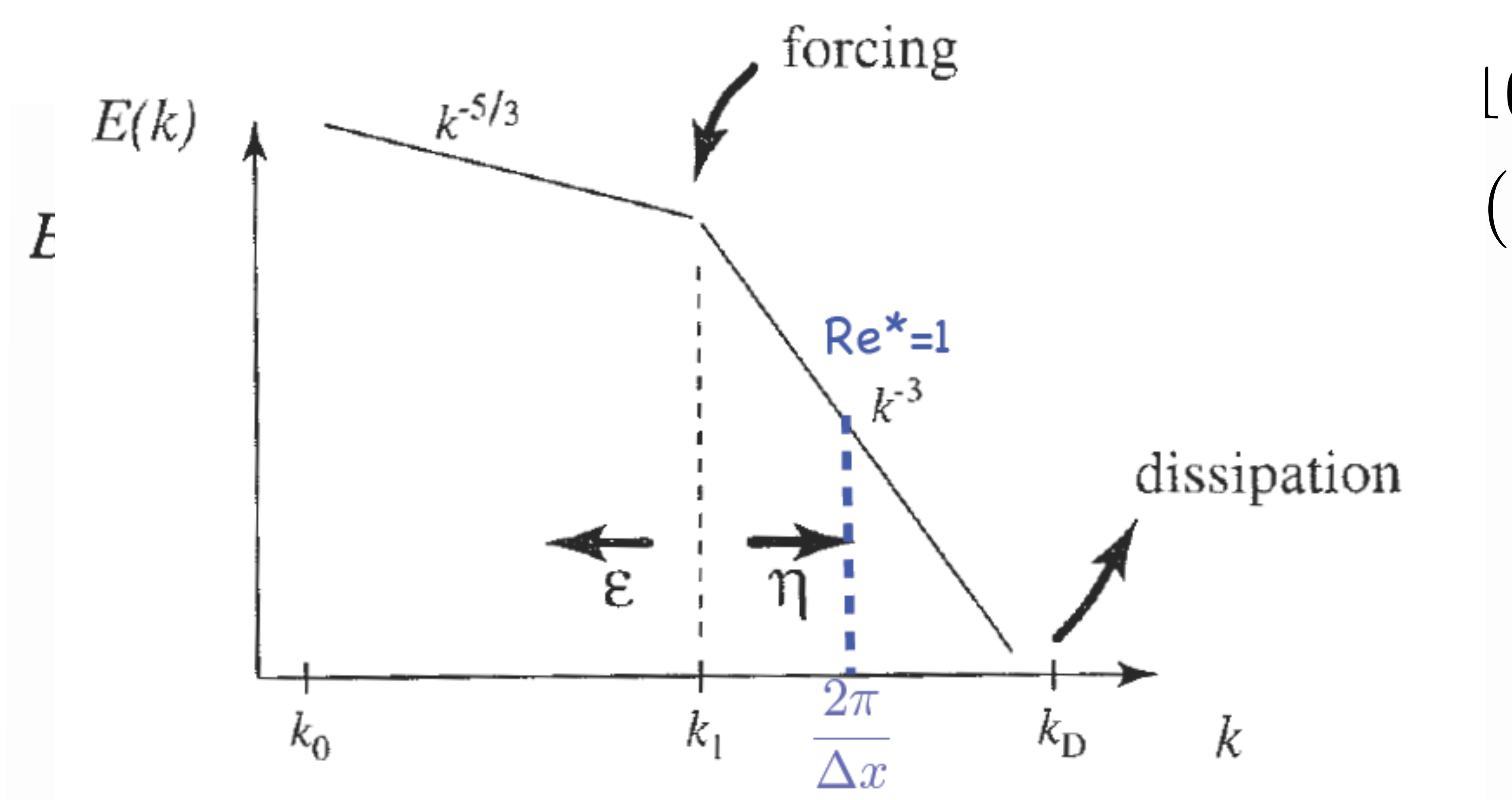

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Navier-Stokes Equations: Rotationally Constrained Flows, $Ro \ll 1$

- For $Ro \ll 1$ turbulence challenge compounded

$$D_t \boldsymbol{u} + Ro^{-1}\widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu\nabla p + \Gamma b \widehat{\boldsymbol{z}} + Re^{-1}\nabla^2 \boldsymbol{u}$$

$$D_t b - \Gamma^{-1} Fr^{-2} w \partial_z \rho(z) = Pe^{-1}\nabla^2 b$$

$$\nabla \cdot \boldsymbol{u} = 0$$

- NSE a stiff PDE, $\exists$ fast inertial waves & slow geostrophically balanced eddies

Fast Inertial Waves

$$\omega_{fast} \sim Ro^{-1}\frac{k_z}{\sqrt{k_\perp^2 + k_z^2}}$$

of secondary importance

Geostrophic Eddies/Slow Waves

$$\omega_{slow} \sim \mathcal{O}(1)$$

$$Ro^{-1}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0 \Rightarrow$$

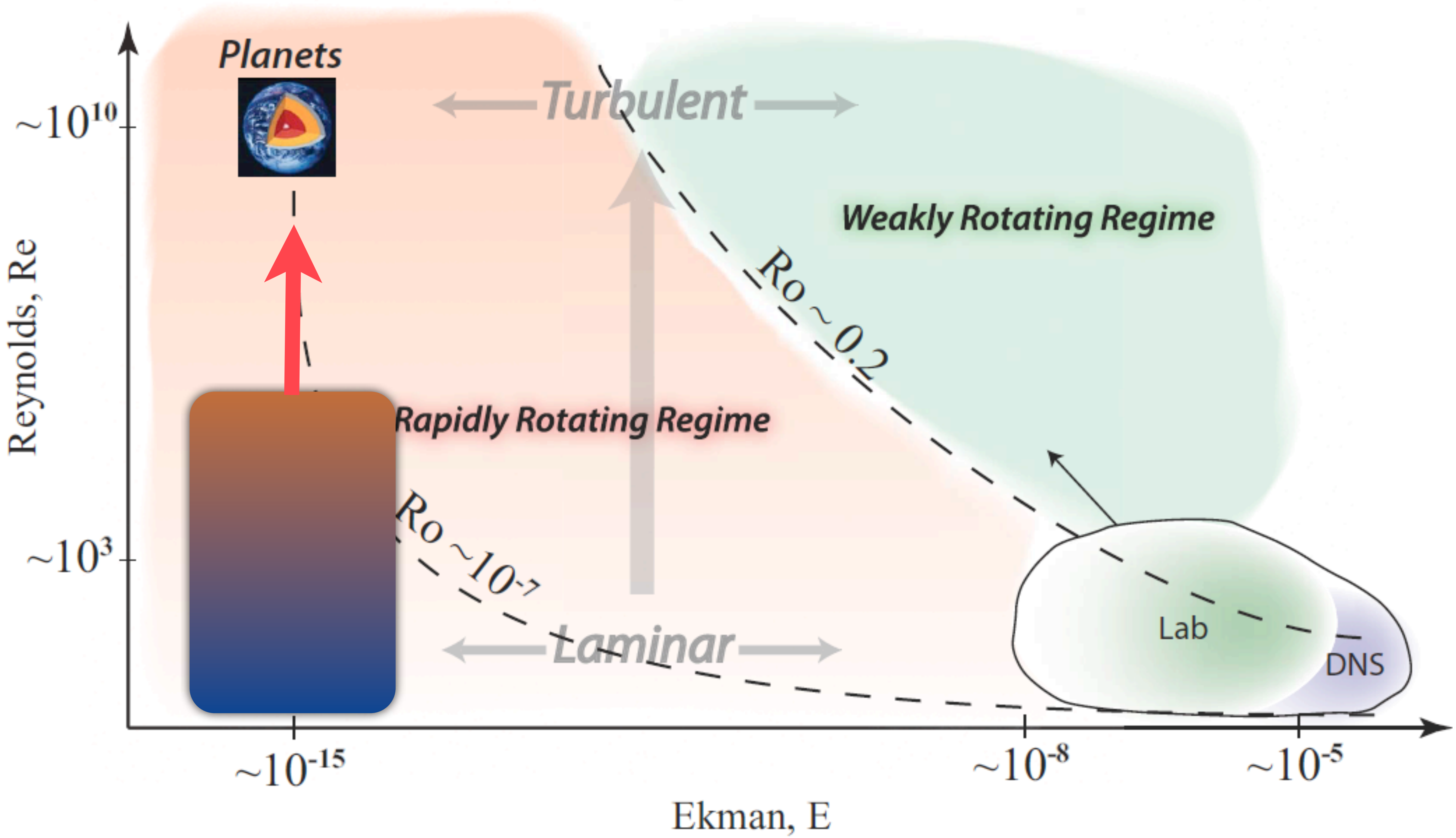
$$\widehat{\boldsymbol{z}} \cdot \nabla(\boldsymbol{u}, p) \approx 0$$

Proudman-Taylor Thm (1916,1923)

Low Rossby Number Challenge

- Fast waves + geostrophically balanced eddies limit DNS/Lab investigations

Resolution: Quasi-Geostrophic Theory. Restrict dynamics to geostrophic manifold and identify Reduced (Nonhydrostatic) PDE's!



Reduced QG Equations: Perturbation Theory

- Select aspect ratio of interest, set distinguished limits

- Perform asymptotic expansion in Rossby number, Ro <<1

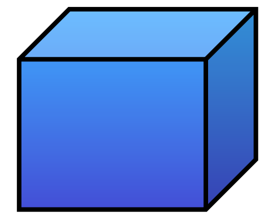
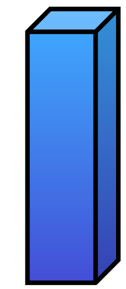
$$u = u_0 + Ro\, u_1 + Ro^2\, u_2 + \cdots$$
$$v = v_0 + Ro\, v_1 + Ro^2\, v_2 + \cdots$$
$$\vdots$$

- Projection to slow manifold J. et al JFM 06, J. & Knobloch JMP 07, Calkins et al JFM 13

- Solve sequence of LPDE's with secularity conditions

Rotationally constrained flows and aspect ratio

$Ro \ll 1$ $\qquad$ $A = H/L$

QG	Intermediate	Convection
		
$H/L << 1$	$H/L = O(1)$	$H/L >> 1$
Charney (1948)	Embid & Majda (1998)	J. et al TCFD '98; JFM 06

- Unified QG approach:

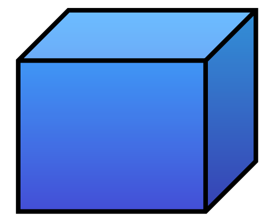
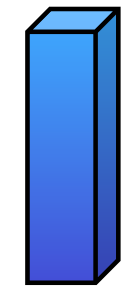
pointwise geostrophy: $Ro^{-1}\widehat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp} = -Eu\nabla_{\perp} p$

inc. vortex stretching: $\frac{U^*}{L} \cdot \frac{U^*}{L} \sim \frac{2\Omega W^*}{H}$

vert. velocity scaling: $w_0 = \mathcal{O}(ARo)$

Rotationally constrained flows and aspect ratio

$Ro \ll 1$ $\qquad$ $A = H/L$

QG	Intermediate	Convection
		
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- Unified QG approach: $\partial_z \to A^{-1}\partial_z, \quad w = ARoW$

$$\widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\nabla_\perp p, \;\; p = \psi$$

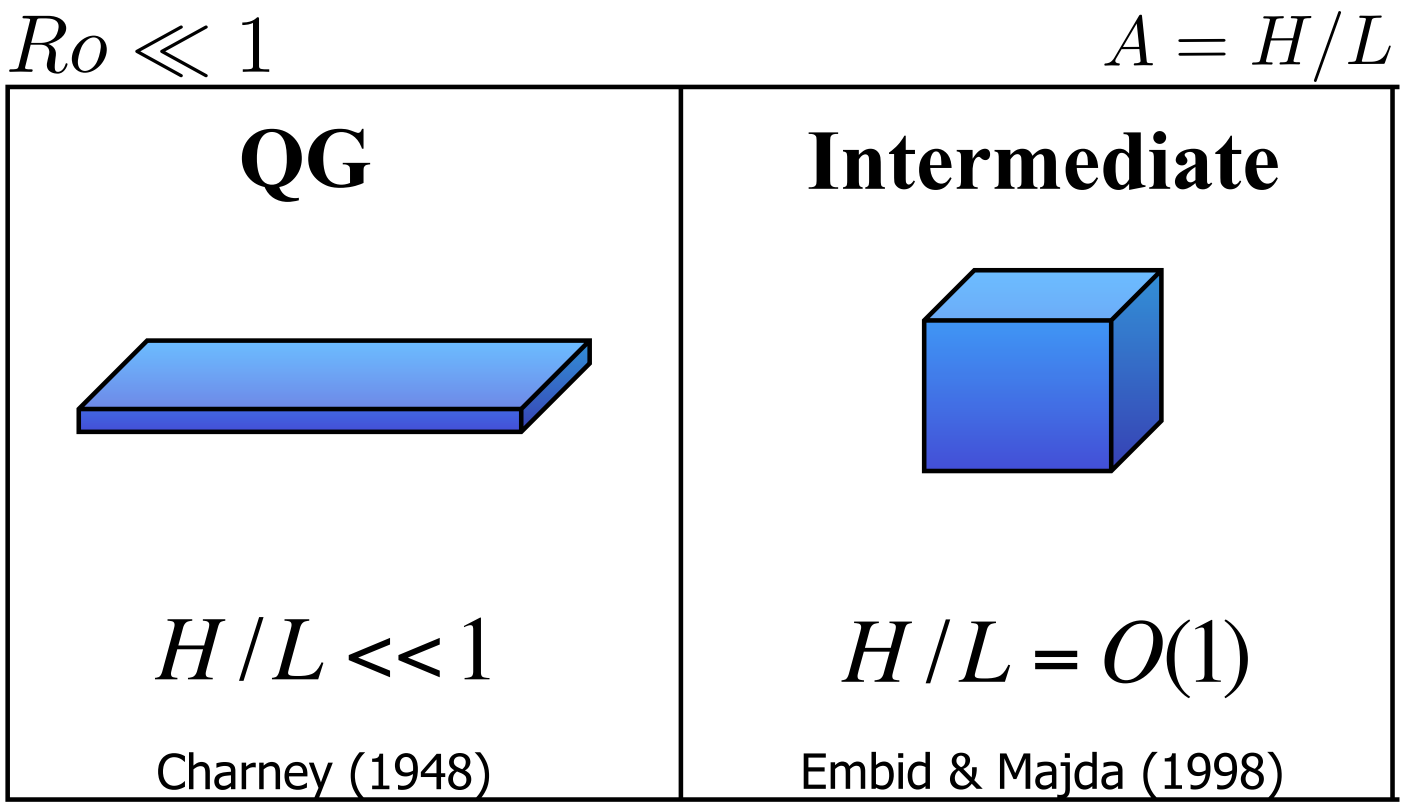
$$D_t^\perp \nabla_\perp^2 \psi - \partial_z W = Re^{-1} \nabla_A^2 \nabla_\perp^2 \psi$$

$$\boxed{(ARo)^2} D_t^\perp W = (-\partial_z \psi + b) + \boxed{(ARo)^2} Re^{-1} \nabla_A^2 W$$

$$D_t^\perp b - W \partial_z \rho(z) = Pe^{-1} \nabla_A^2 b$$

- Unified distinguished limits: $Eu = Ro^{-1}, \;\; \Gamma = (ARo)^{-1}, \;\; Fr = ARo$

H-QG



Conservation: Energy, Enstrophy

PV is the dynamical variable

- Unified QG approach: $\partial_z \to A^{-1}\partial_z, \quad w = ARoW$

$$\widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\nabla_\perp p, \;\; p = \psi$$

$$D_t^\perp \nabla_\perp^2 \psi - \partial_z W = Re^{-1} \nabla_A^2 \nabla_\perp^2 \psi$$

$$0 = (-\partial_z \psi + b)$$

$$D_t^\perp b - W \partial_z \rho(z) = Pe^{-1} \nabla_A^2 b$$

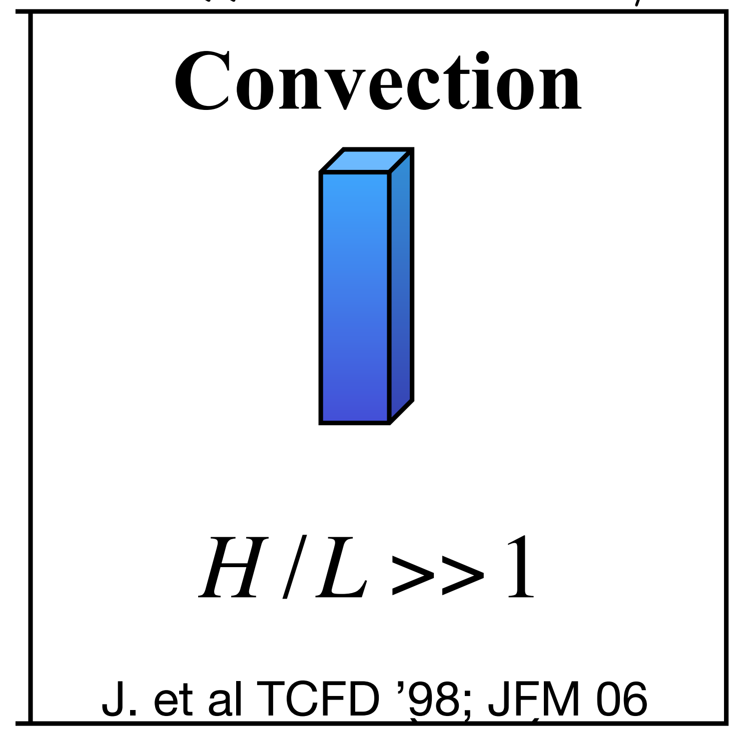
- Unified distinguished limits: $Eu = Ro^{-1}, \;\; \Gamma = (ARo)^{-1}, \;\; Fr = ARo$

NH-QG

Conservation: Energy

PV not a dynamical variable

$Ro \ll 1 \qquad A = H/L$



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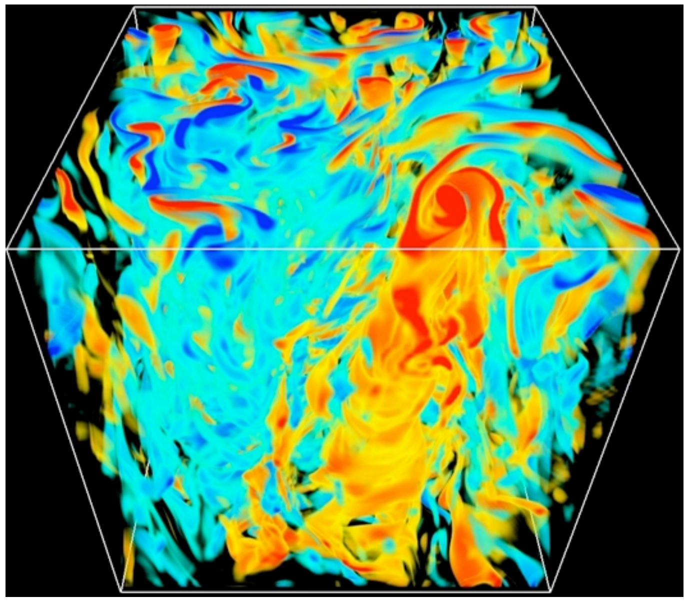
- Unified distinguished limits: $Eu = Ro^{-1}, \;\Gamma = (ARo)^{-1}, \;Fr = ARo$

Application to Thermal Convection

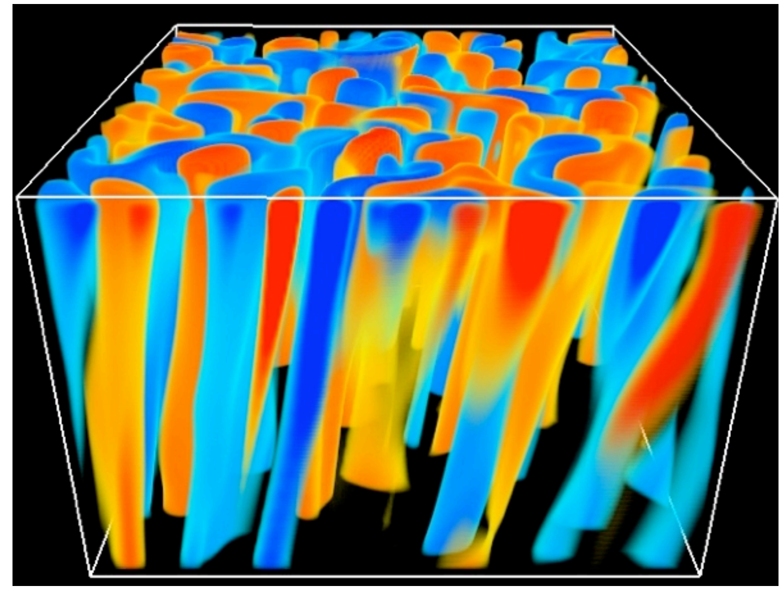
Plane layer convection:

upright turbulent



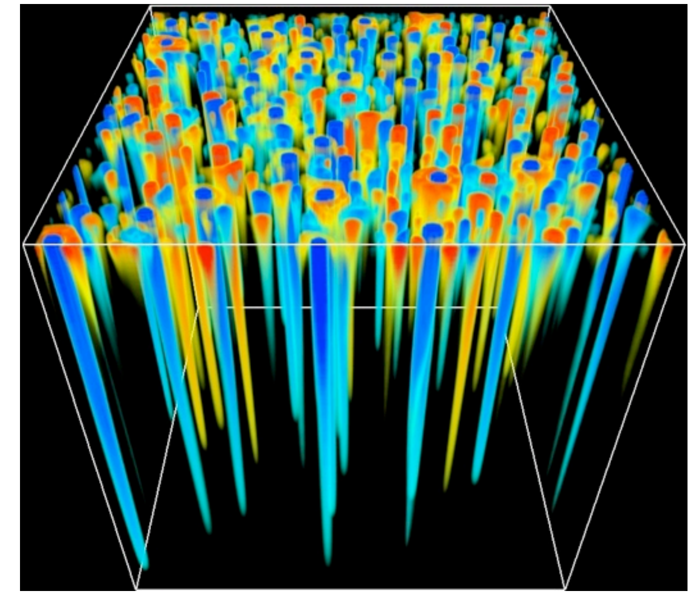
J., Knobloch PoF 99, JMP ‘07

tilted f-plane, columnar



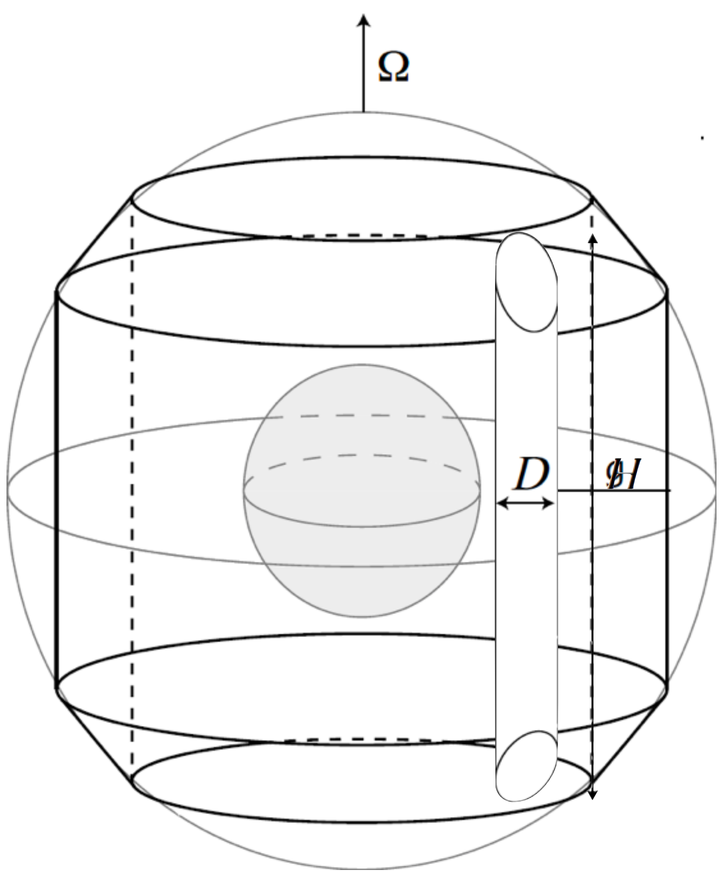
Sprague et al JFM 06, J. et al JFM 06

penetrative

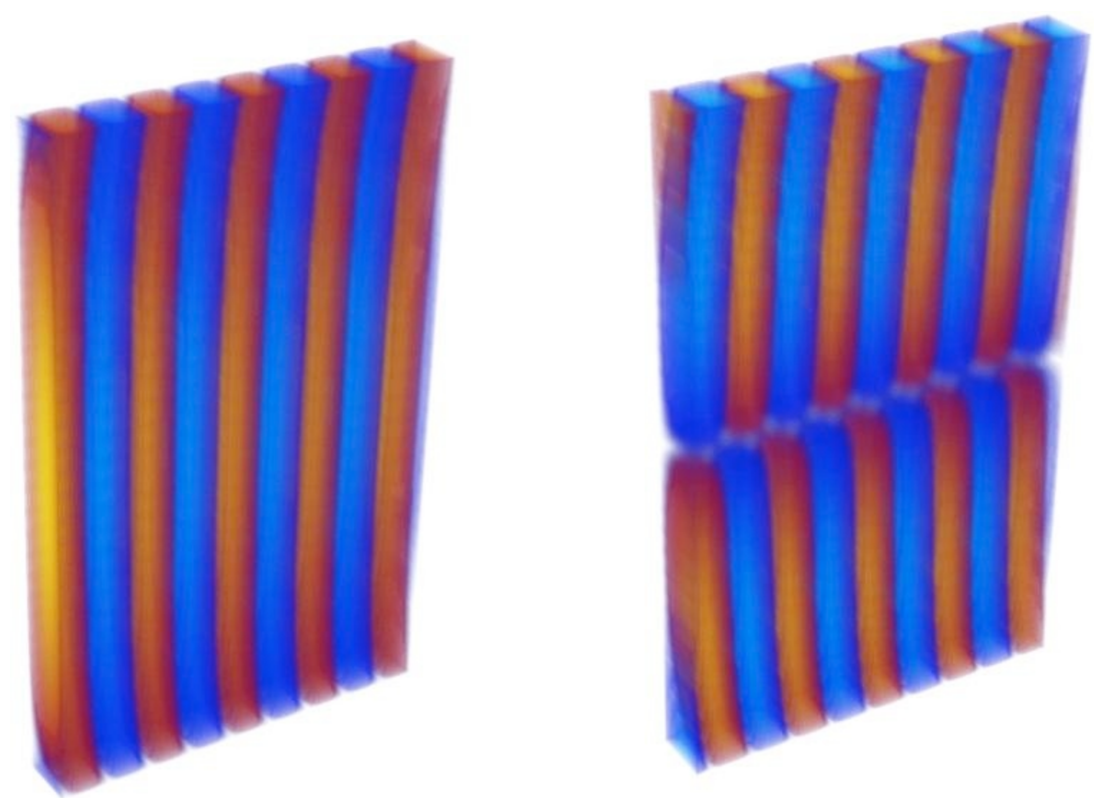


Groom et al PRL ’10, J. et al GAFD 12, J. et al PRL 12

Planetary scale convection:



Calkins et al JFM 13



thermal Rossby waves

QuasiGeostrophic Rayleigh-Bénard Convection

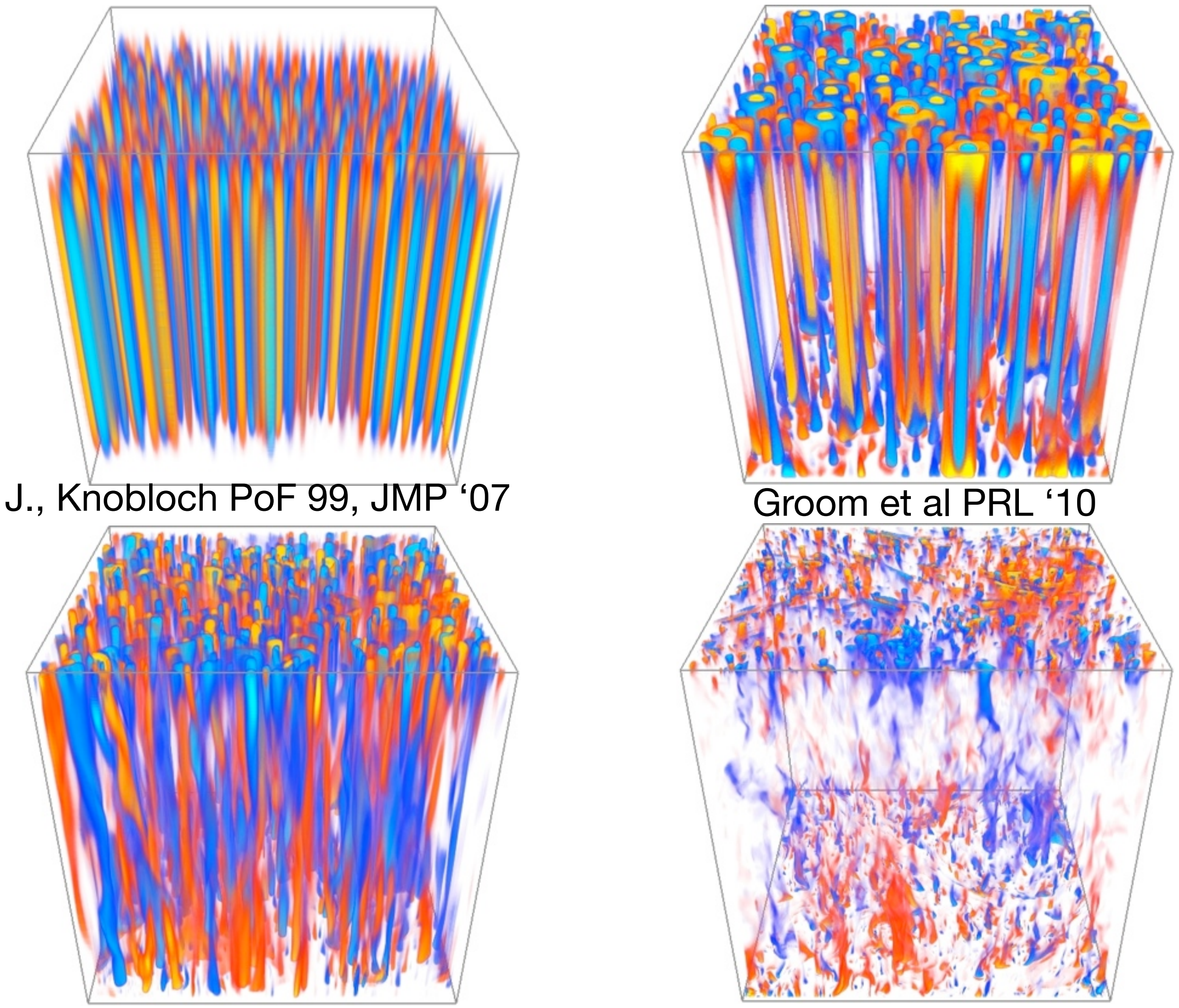
$$\partial_t \zeta + J\left[\psi, \zeta\right] - \partial_Z w = \nabla_\perp^2 \zeta$$

$$\partial_t w + J\left[\psi, w\right] + \partial_Z \psi = \nabla_\perp^2 w + \frac{RaE^{4/3}}{\sigma}\theta$$

$$\partial_t \theta + J\left[\psi, \theta\right] + w\partial_Z \overline{T} = \frac{1}{\sigma}\nabla_\perp^2 \theta$$

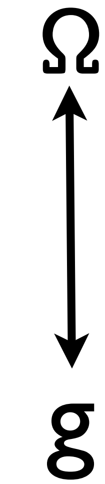
$$\partial_Z \overline{w\theta} = \frac{1}{\sigma}\partial_{ZZ}\overline{T}$$

$RaE^{4/3} = 10, \sigma = 15$ $RaE^{4/3} = 40, \sigma = 15$



J., Knobloch PoF 99, JMP '07 Groom et al PRL '10

$RaE^{4/3} = 40, \sigma = 1$ $RaE^{4/3} = 100, \sigma = 1$



- Four Flow Regimes as Ra ↑ : laminar to turbulent

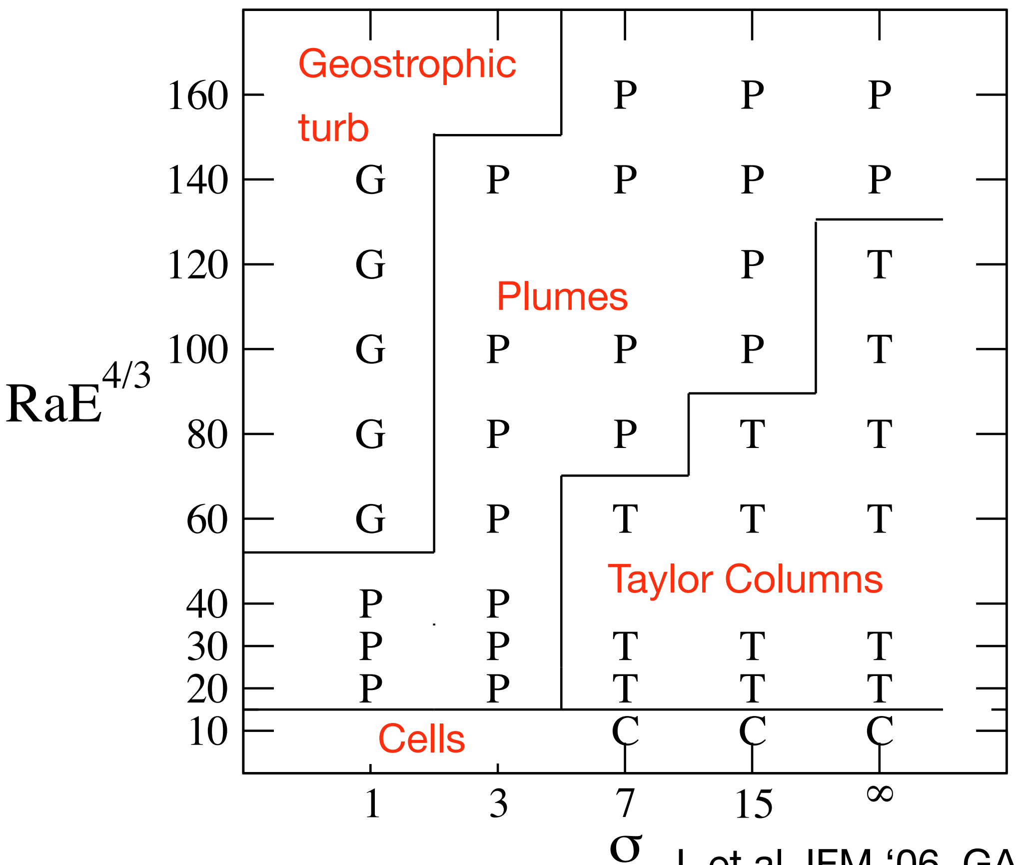
  Cells → CTC's via TBL instability & synchronization of TBL's

  CTC's: Shielded vortical columns with zero circulation

  Plumes regime occurs when TBL are unable to synchronize

- Ultimate Regime Geostrophic Turbulence (Julien et al GAFD 2012)

QuasiGeostrophic Rayleigh-Bénard Convection



J. et al JFM '06, GAFD '12

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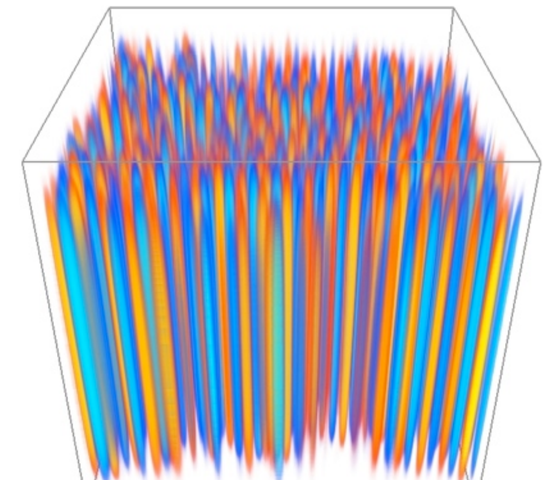

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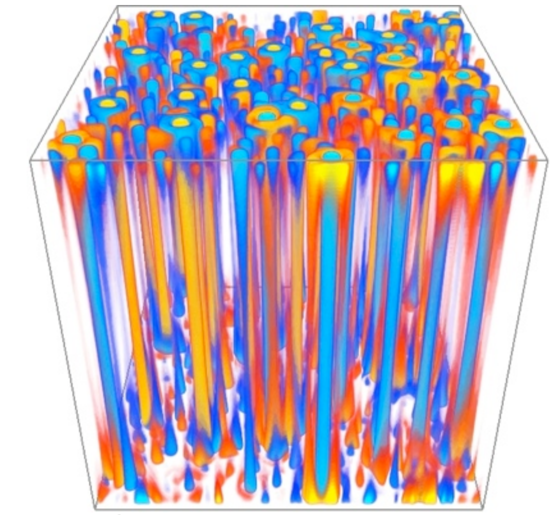
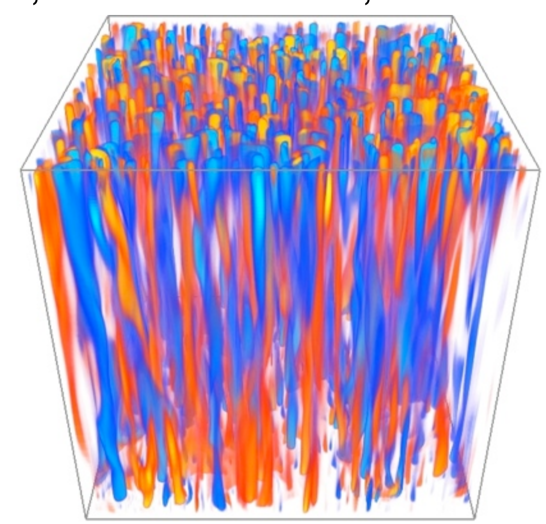
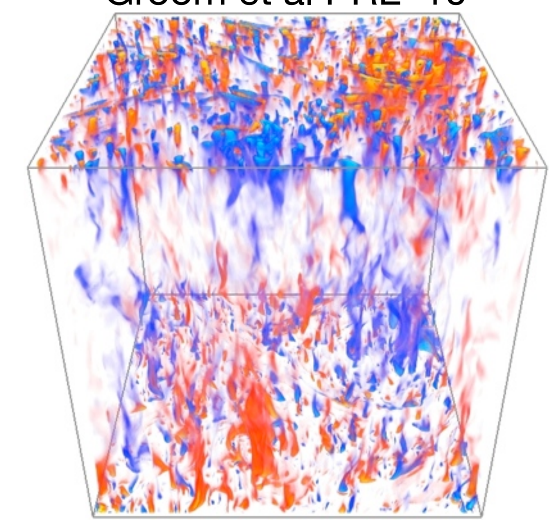


Groom et al PRL '10

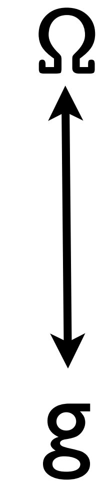


$RaE^{4/3}$ = 40, σ = 1



$RaE^{4/3}$ = 100, σ = 1

J. et al GAFD '12



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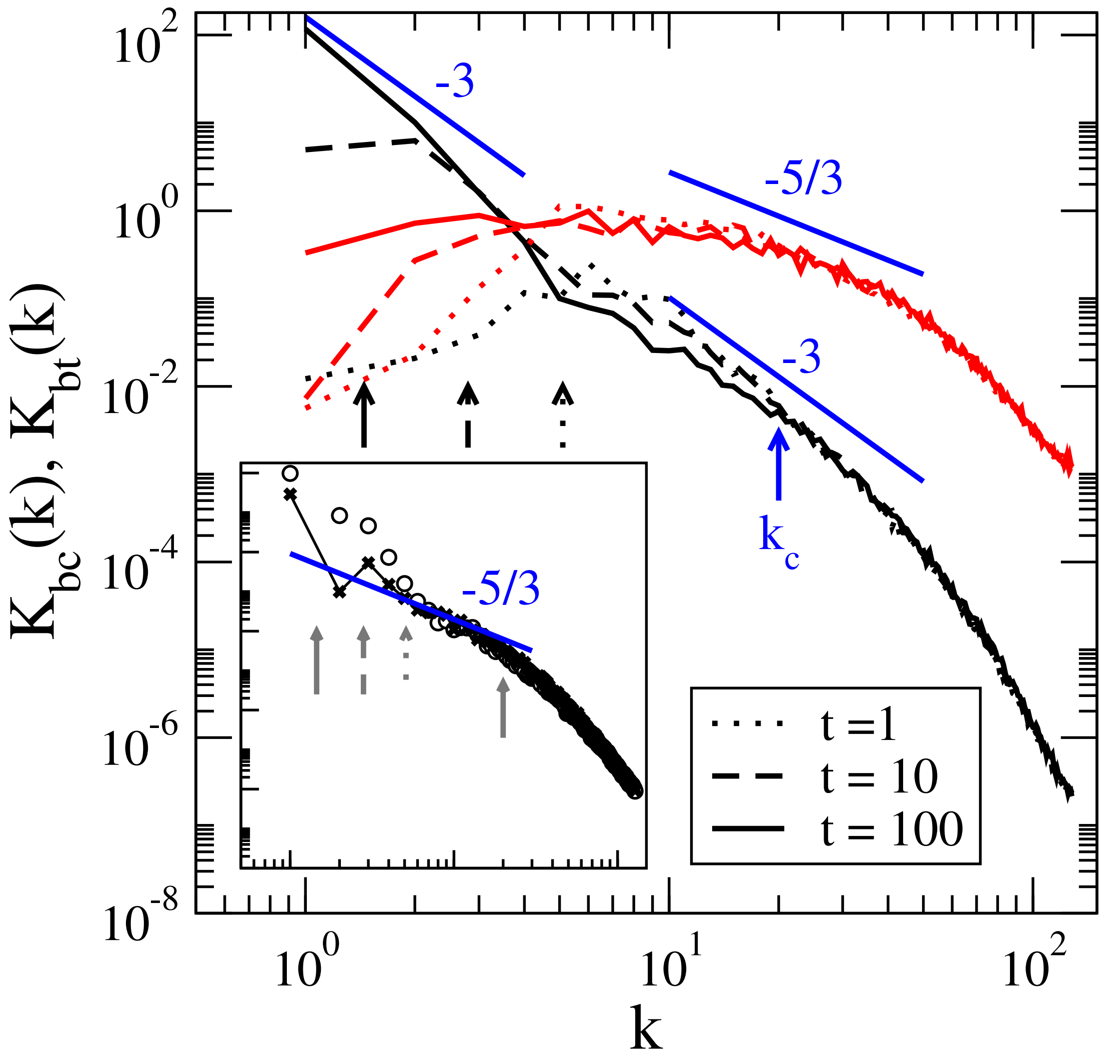
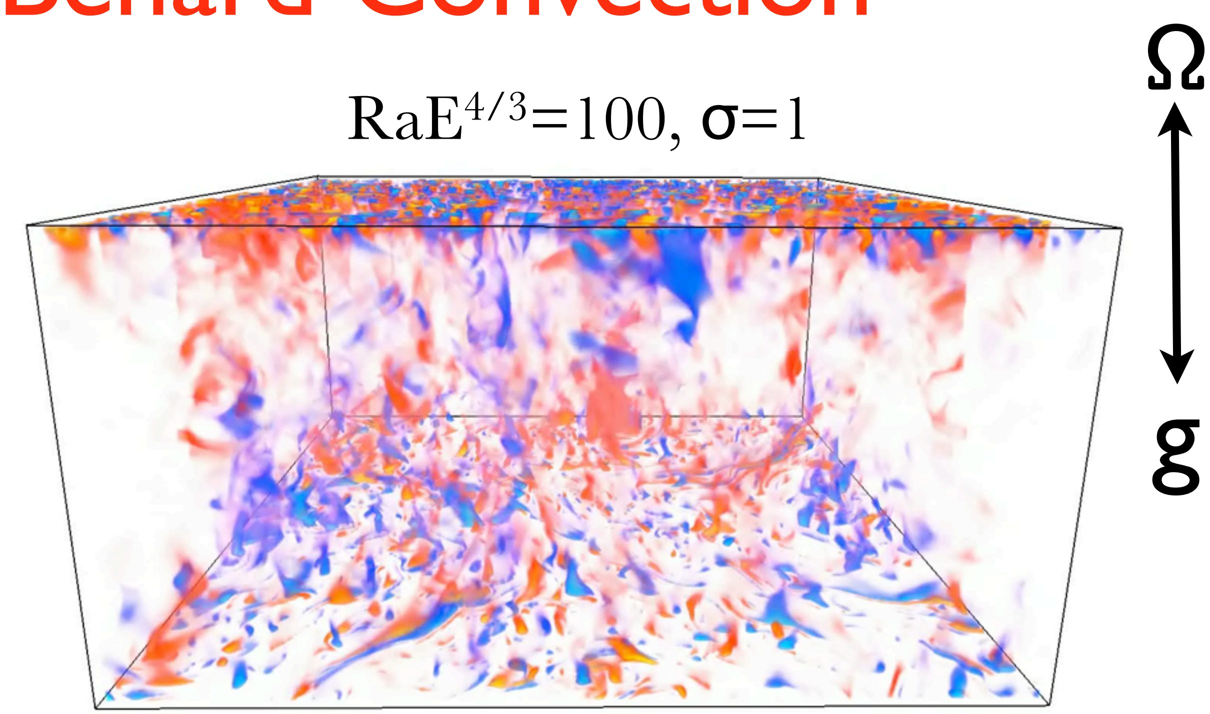
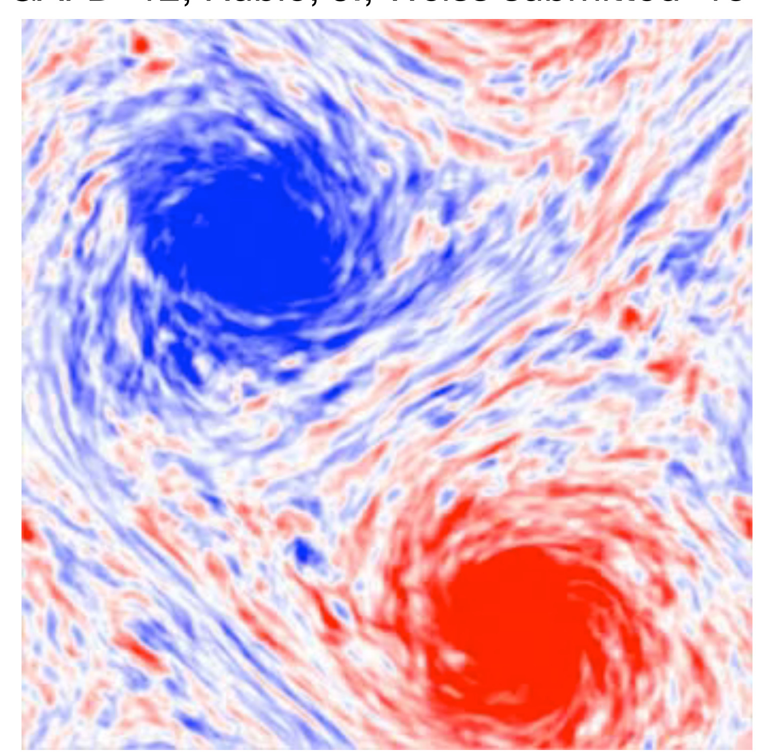
Geostrophic Rayleigh-Bénard Convection





J. et al GAFD '12; Rubio, J., Weiss submitted '13



Depth averaged vorticity

- Turbulent Inverse Cascade (J. et al GAFD '12, Rubio et al 2014)

  Positive feedback loop

  - GT provides the nonlinear forcing that generates BV's

  - BV organizes GT thru advection and stretching

  - BV produces large scale forcing to sustain itself

- Energy spectra consistent with 2D barotropic and 3D baroclinic dynamics

Heat Transport by GT Convection $Nu - 1 = \frac{1}{25}\sigma^{-\frac{1}{2}}\left(RaE^{\frac{4}{3}}\right)^{\frac{3}{2}}$

- Thermal throttling: region that controls the efficiency of heat transport in the fluid layer. TBL ? or Interior?

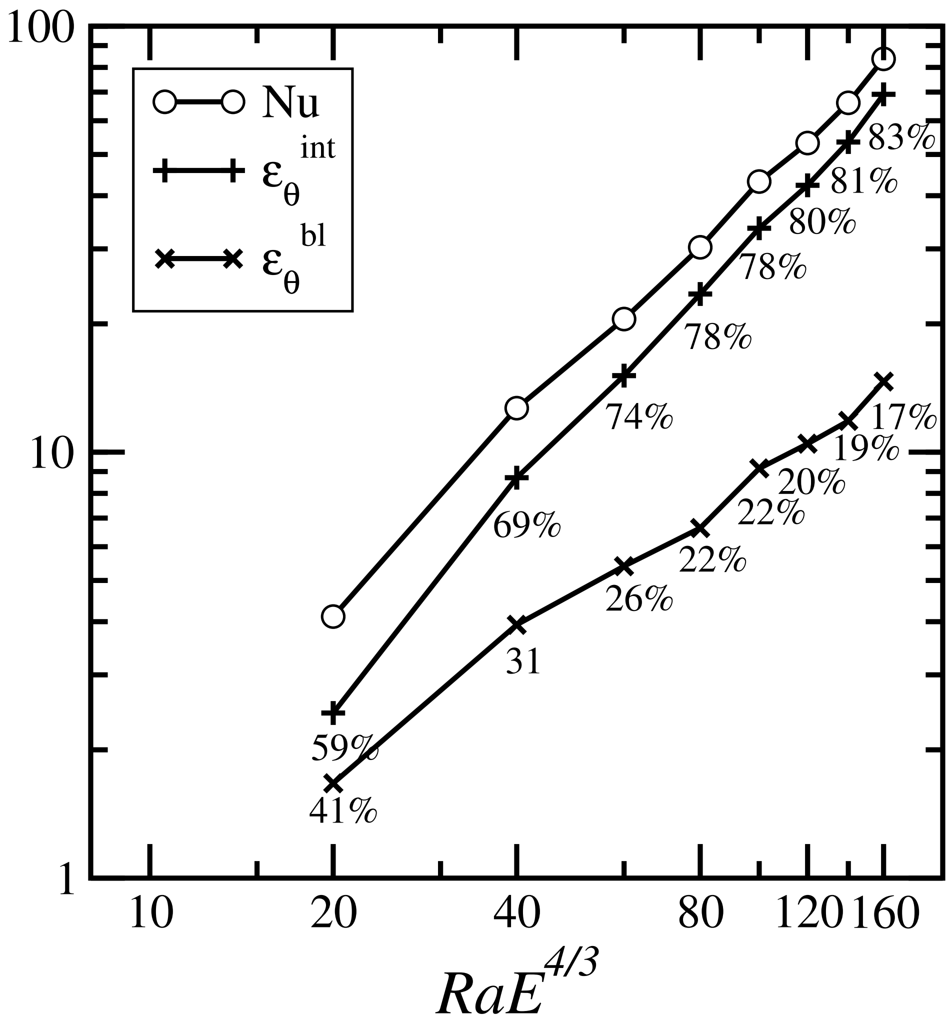
turbulent interior $\beta \Rightarrow$ dissipationless scaling law (Kraichnan '63, Howard '63)

TBL $\beta \Rightarrow$ marginally stably BL's (Malkus '63)

$\beta_{turb} = 3/2$

$\beta_{tbl} = 3$

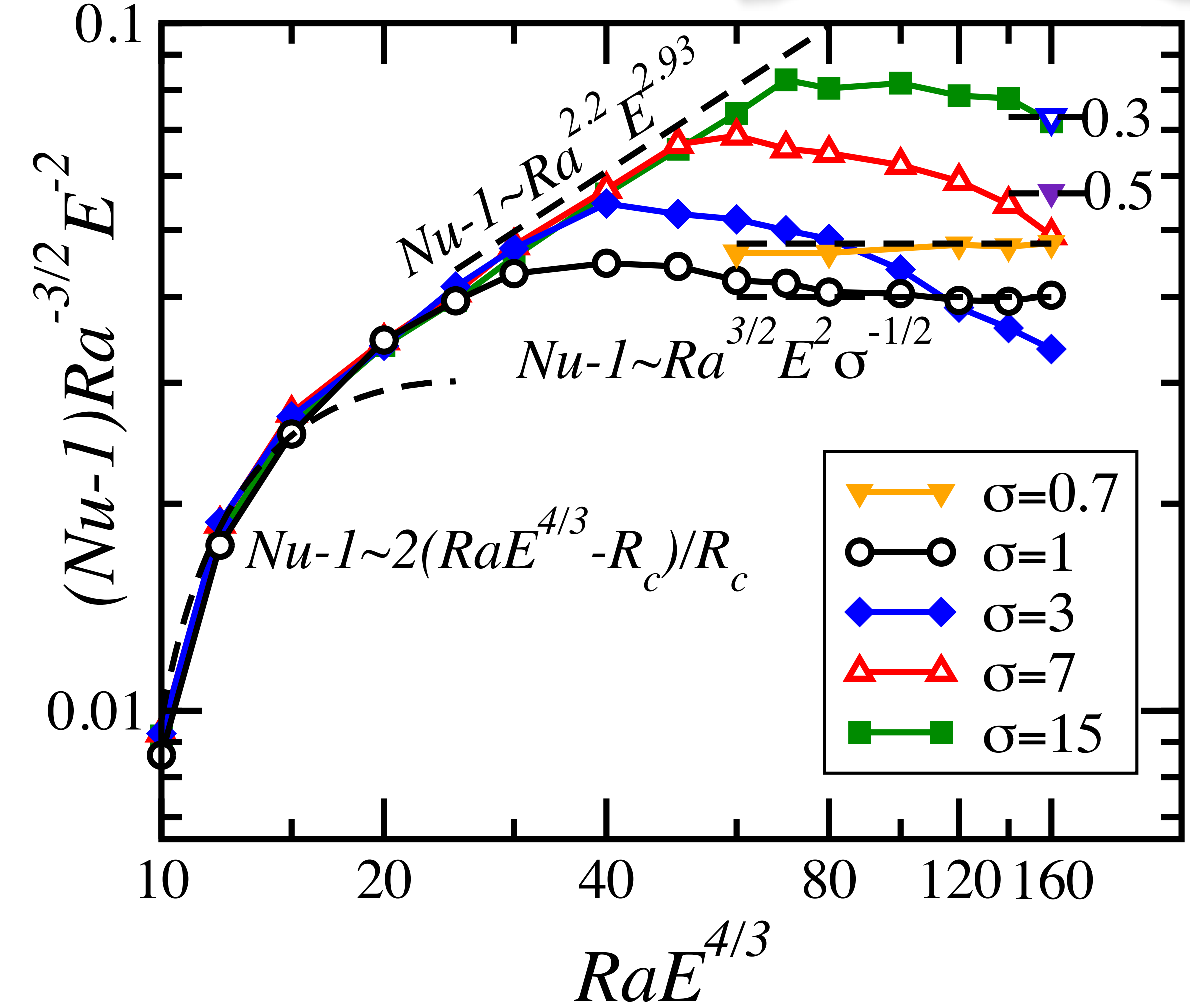
rotating



$$\mathcal{E}_\theta \approx \mathcal{E}_\theta^{int} = \left\langle \left|\partial_Z \overline{T}\right|^2 \right\rangle + \left\langle |\nabla_\perp \theta|^2 \right\rangle$$

$$\equiv Nu$$

J. et al PRL '12



$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g\alpha\Delta T H^3}{\nu\kappa}, \quad E = \frac{\nu}{fH^2}$$

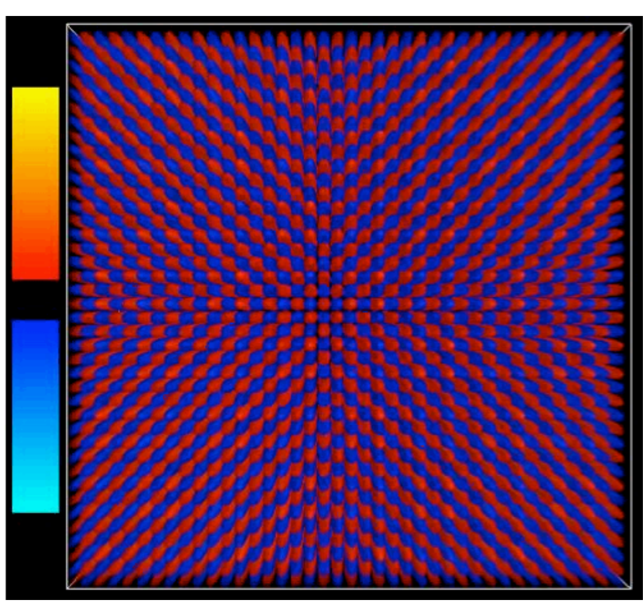
$$\sigma = \frac{\nu}{\kappa} \qquad Ra_{crit} \propto E^{-4/3}$$

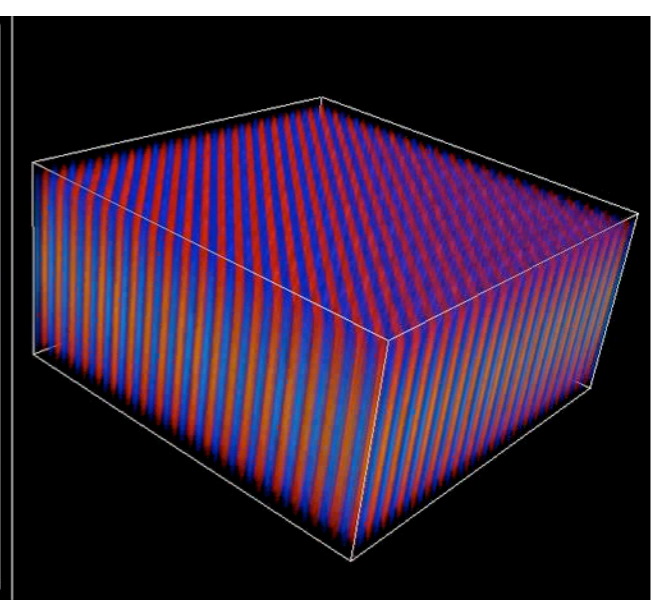
Outlook for 3D NH-QG *Thank you*

- Reduced PDE’s well suited to NHQG dynamics, computationally less challenging.
- Incompressible/Anelastic aDNS (“a”symptotic)
  - Investigate route to turbulence
  - Mean flow generation: inverse turbulent cascade?
  - Efficiency of heat transport: turbulent scaling laws
- ? Multiscale modeling: Coupling to balanced large-scale dynamics

  Grooms, Fox-Kemper & J DAO ’11
- Planetary convection: deep spherical shells

Sprague et al JFM ’06 Groom et al PRL ‘10





Julien et al GAFD ‘12 Julien et al PRL ‘12