Asymptotic Approaches for High Reynolds Number Shear Flows

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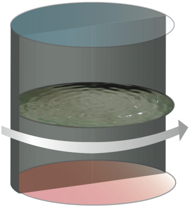
UNH Workshop on High-*Re* Wall Flows
November 21st, 2013

From Constrained Convection to Wall-Bounded Shear Flows

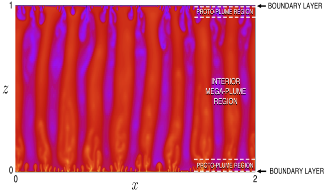
I: Anisotropic Driving
\+ Linear Instability
\+ External Constraint



E. King

Rapidly rotating convection

II: Anisotropic Driving
\+ Linear Instability



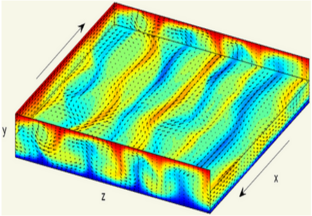
D. Hewitt

Porous medium convection

S. Monismith

Langmuir circulation

III: Anisotropic Driving



J. Gibson

Plane Couette flow

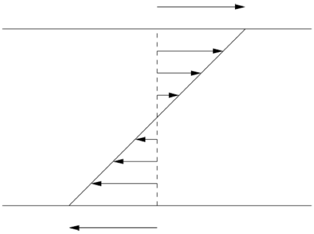
Asymptotic Reduction for High-*Re* Wall Flows?

Relation to Workshop Themes & Focus Questions:

1. Illustration of model reduction enabled by flow structure/anisotropy in extreme parameter regimes yet in (apparent) **absence** of externally imposed constraints
2. Reduced model confirms that **exact coherent states (ECS)** are **not** limited to transitional flows, but persist as $Re \to \infty$, and reveals structure in that limit
3. Dynamics are **quasi-linear** about streamwise-averaged streamwise flow that is **self-consistently** determined by **nonlinear** processes
4. Reduced formulation may provide **theoretical framework** for understanding ECS in **spatially-extended** domains at large Re?

Testbed for Asymptotically-Reduced Modeling of Wall-Bounded Shear Flows

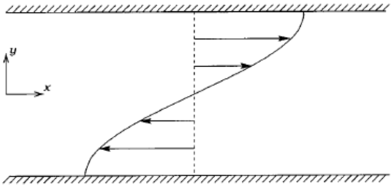
Plane Couette Flow (PCF)



Wall BCs: $u \overset{+}{\underset{-}{=}} 1, \; v = w = 0$

Forcing: $\mathbf{f}(y) = \mathbf{0}$

Waleffe Flow



Wall BCs: $\partial_y u = 0, \; v = w = 0$

Forcing: $\mathbf{f}(y) = \frac{\sqrt{2}\pi^2}{4Re} \sin\left(\frac{\pi y}{2}\right) \hat{\mathbf{e}}_x$

Incompressible Navier–Stokes (NS) Equations

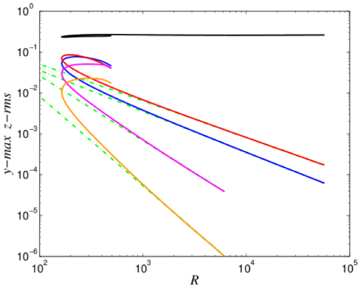
$$\frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} + \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = 0 \qquad Re = UH/\nu$$

Basis for Asymptotically Reduced PDE Model of Lower-Branch ECS in PCF

Inspiration: Notion that **streamwise rolls** *weak* compared to **streamwise streaks** (roughly complementary scenario relative to "pure" Langmuir turbulence)

Justification: Wang, Gibson & Waleffe, *PRL* (2007)



Fourier decomposition for steady-state ECS:

$$\mathbf{u}(x,t) = \sum_{n=-\infty}^{n=+\infty} \hat{\mathbf{u}}_n(y,z)e^{in\alpha x}$$

Scalings:

- $\hat{u}_0 = O(1)$
- $(\hat{v}_0, \hat{w}_0) = O(Re^{-1})$
- $\hat{\mathbf{u}}_1 = O(Re^{-0.9})$
- $\hat{\mathbf{u}}_n = o(Re^{-1})$ for $n > 1$

Large-Re Multiple Scale Asymptotic Analysis

- Identify $\epsilon \equiv 1/Re$, where $\epsilon \ll 1$
- Introduce slow streamwise length scale $X \equiv \epsilon x$ and slow time scale $T = \epsilon t$ s.t. $\partial_x \to \partial_x + \epsilon \partial_X$ and $\partial_t \to \partial_t + \epsilon \partial_T$
- Decompose $(\mathbf{v}, p) = (\bar{\mathbf{v}}, \bar{p})(X, y, z, T) + (\mathbf{v}', p')(x, X, y, z, t, T)$, where $\overline{(\cdot)}$ = fast-(x,t) average and $(\cdot)'$ = fluctuation about mean
- Define $\mathbf{v} = u\hat{\mathbf{e}}_\mathbf{x} + \mathbf{v}_\perp$ and expand

$$
\boxed{
\begin{aligned}
u &\sim \bar{u}_0 + \epsilon\left(\bar{u}_1 + u_1'\right) + \ldots \\
\mathbf{v}_\perp &\sim \epsilon\left(\bar{\mathbf{v}}_{1\perp} + \mathbf{v}_{1\perp}'\right) + \ldots \\
p &\sim \epsilon\left(\bar{p}_1 + p_1'\right) + \epsilon\left(\bar{p}_2 + p_2'\right) + \ldots
\end{aligned}
}
$$

- Substitute into NS equations and fast-average to eliminate secular growth terms

Large-*Re* Reduced PDE Model [cf. SSP of Waleffe (1995,1997); VWI of Hall (1991,2010)]

Mean Equations

$$
\begin{aligned}
\partial_T \bar{u}_0 + \bar{u}_0 \partial_X \bar{u}_0 + \left(\bar{\mathbf{v}}_{1\perp} \cdot \nabla_\perp\right) \bar{u}_0 &= \nabla_\perp^2 \bar{u}_0 + f(y) \\
\partial_T \bar{\mathbf{v}}_{1\perp} + \partial_X \left[\bar{u}_0 \bar{\mathbf{v}}_{1\perp}\right] + \nabla_\perp \cdot \left[\bar{\mathbf{v}}_{1\perp} \bar{\mathbf{v}}_{1\perp} + \overline{\mathbf{v}'_{1\perp} \mathbf{v}'_{1\perp}}\right] &= -\nabla_\perp \bar{p}_2 + \nabla_\perp^2 \bar{\mathbf{v}}_{1\perp} \\
\partial_X \bar{u}_0 + \nabla_\perp \cdot \bar{\mathbf{v}}_{1\perp} &= 0
\end{aligned}
$$

Large-*Re* Reduced PDE Model [cf. SSP of Waleffe (1995,1997); VWI of Hall (1991,2010)]

Mean Equations

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\partial_T \bar{u}_0 + \bar{u}_0 \partial_X \bar{u}_0 + (\bar{\mathbf{v}}_{1\perp} \cdot \nabla_\perp) \bar{u}_0 &= \nabla_\perp^2 \bar{u}_0 + f(y) \\
\partial_T \bar{\mathbf{v}}_{1\perp} + \partial_X [\bar{u}_0 \bar{\mathbf{v}}_{1\perp}] + \nabla_\perp \cdot \left[\bar{\mathbf{v}}_{1\perp} \bar{\mathbf{v}}_{1\perp} + \overline{\mathbf{v}'_{1\perp} \mathbf{v}'_{1\perp}} \right] &= -\nabla_\perp \bar{p}_2 + \nabla_\perp^2 \bar{\mathbf{v}}_{1\perp} \\
\partial_X \bar{u}_0 + \nabla_\perp \cdot \bar{\mathbf{v}}_{1\perp} &= 0
\end{aligned}$$

Fluctuation Equations

$$\begin{aligned}
\partial_t u'_1 + \bar{u}_0 \partial_x u'_1 + (\mathbf{v}'_{1\perp} \cdot \nabla_\perp) \bar{u}_0 &= -\partial_x p'_1 \\
\partial_t \mathbf{v}'_{1\perp} + \bar{u}_0 \partial_x \mathbf{v}'_{1\perp} &= -\nabla_\perp p'_1 \\
\partial_x u'_1 + \nabla_\perp \cdot \mathbf{v}'_{1\perp} &= 0
\end{aligned}$$

Structure of Large-*Re* Reduced PDE Model

- In absence of X-modulation, mean system is **2D** but **3C** and has **unit** effective Reynolds number
- Departure from base laminar flow driven entirely by $\overline{\mathbf{v}'_{1\perp}\mathbf{v}'_{1\perp}}$ Reynolds stress
- Fluctuation equations are: (i) **inviscid**; (ii) **quasi-linear** $\Rightarrow$ admit **single-mode solutions** in x, e.g.

$$\mathbf{v}'_{1\perp} = A\hat{\mathbf{V}}_{1\perp}(y,z)e^{i(\alpha x)} + c.c.$$

and (iii) **singular** for equilibrium ECS on **non-planar critical layer** $\bar{u}_0(y,z) = 0$

$$\nabla^2_\perp \hat{P}_1 - \alpha^2 \hat{P}_1 - \frac{2}{\bar{u}_0}\nabla_\perp \bar{u}_0 \cdot \nabla_\perp \hat{P}_1 = 0$$

Generalized Rayleigh equation

Hall & Horseman, *JFM* (1991)

Viscous Regularization of CL. I. Composite Equation [Beaume *et al.* (2012,2013)]

Eigenvalue Formulation of Regularized Fluctuation Equations

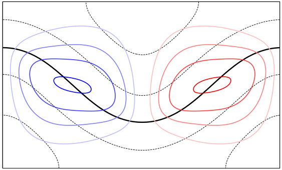
$$\begin{aligned} \nabla_\perp^2 \hat{P}_1 - \alpha^2 \hat{P}_1 &= -2i\alpha\left[\hat{V}_1 \partial_y \bar{u}_0 + \hat{W}_1 \partial_z \bar{u}_0\right] \\ \sigma \hat{V}_1 + i\alpha \bar{u}_0 \hat{V}_1 &= -\partial_y \hat{P}_1 + \epsilon \nabla_\perp^2 \hat{V}_1 \\ \sigma \hat{W}_1 + i\alpha \bar{u}_0 \hat{W}_1 &= -\partial_z \hat{P}_1 + \epsilon \nabla_\perp^2 \hat{W}_1 \end{aligned}$$

Solution Algorithm

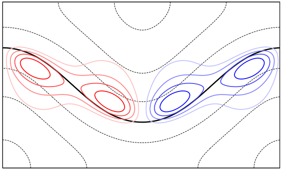
1. Generate initial guess for $\bar{u}_0(y, z)$
2. Guess amplitude of fluctuations A
3. Solve eigenvalue problem (via Arnoldi iteration) for form of fastest-growing (non-oscillatory) modes $\hat{V}_1(y, z)$, $\hat{W}_1(y, z)$
4. Use $A(\hat{V}_1, \hat{W}_1)$ to compute Reynolds stress in mean x-vorticity ($\bar{\Omega}_1$) equation
5. Time-advance $\bar{\Omega}_1$ and $\bar{u}_0$ to steady state
6. Return to step 3. and iterate until convergence
7. Return to step 2., adjusting A until **equilibrium** solution found ($\sigma \equiv 0$)

Viscous Regularization of CL. I. Results for Waleffe Flow: Lower-Branch ECS ($\epsilon^{-1} = 1500$, $\alpha = 0.5$, $L_y = \pi$)

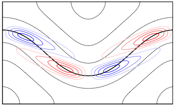
$\bar{\psi}_1(y, z)$



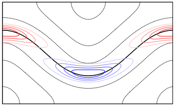
$\bar{\Omega}_1(y, z)$



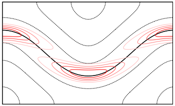
$\mathrm{Re}\{\hat{V}_1(y, z)\}$



$\mathrm{Re}\{\hat{W}_1(y, z)\}$

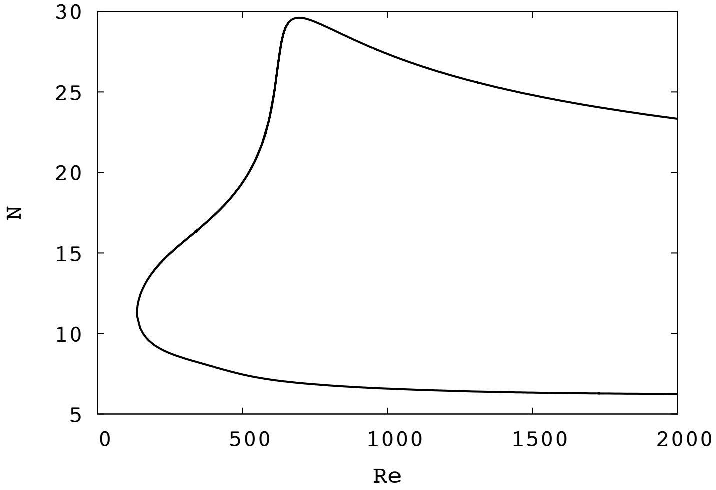


$\hat{V}_1^2(y, z) + \hat{W}_1^2(y, z)$



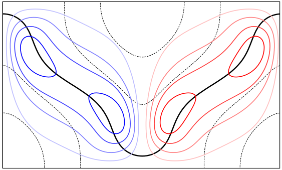
Viscous Regularization of CL. I. Results for Waleffe Flow: Bifurcation Diagram

A surprise. . .

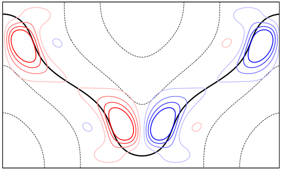


Viscous Regularization of CL. I. Results for Waleffe Flow: Upper-Branch ECS ($\epsilon^{-1} = 1500$, $\alpha = 0.5$, $L_y = \pi$)

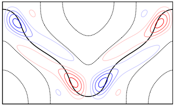
$\bar{\psi}_1(y, z)$



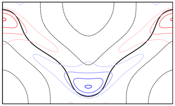
$\bar{\Omega}_1(y, z)$



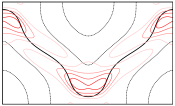
$\mathrm{Re}\{\hat{V}_1(y, z)\}$



$\mathrm{Re}\{\hat{W}_1(y, z)\}$



$\hat{V}_1^2(y, z) + \hat{W}_1^2(y, z)$



Viscous Regularization of CL. II. Matched Asymptotic Analysis of Critical Layer

Hall & Sherwin, *JFM* (2010); Malecha *et al.* (2013)

- Convenient to use $[u \equiv \bar{u}_0(y,z), z]$ coordinates so, e.g., $\boxed{\hat{P}_1(y,z) = \tilde{P}_1(u,z)}$

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- **CL thickness** set by fluctuation dynamics: $\boxed{\epsilon \partial_u^2 \tilde{W}_s - i\alpha u \tilde{W}_s \sim -\partial_y u \partial_z \tilde{P}_1 / |\nabla_\perp u|}$

$\Rightarrow$ $\boxed{u = O(\epsilon^{1/3})}$, consistent with Wang *et al.* (2007)

$\Rightarrow$ Fluctuation fields $(\tilde{V}_1, \tilde{W}_1)$ can be **analytically** related to $\partial_z \tilde{P}_1$ on $u = 0$

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  $\Rightarrow$ Fluctuation fields $(\tilde{V}_1, \tilde{W}_1)$ can be **analytically** related to $\partial_z \tilde{P}_1$ on $u = 0$
- **Fluctuation magnitude** set by balance within CL b/w Reynolds stress forcing by fluctuations and diffusion of mean x-vorticity, *viz.* in Cartesian (y,z) coordinates:

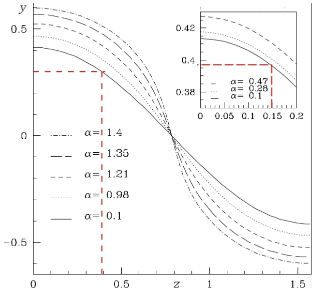
$$\boxed{\nabla_\perp^2 \bar{\Omega}_1 \sim -\left(\partial_z^2 - \partial_y^2\right) \overline{v_1' w_1'} - \partial_z \left[\partial_y \left(\overline{v_1' v_1'} - \overline{w_1' w_1'}\right)\right]}$$

$\Rightarrow$ Find $\boxed{(v', w') = O(Re^{-5/6})}$ w/in CL $\Rightarrow$ Forcing **localized** w/in CL

Viscous Regularization of CL. II. PCF Results as $\alpha \to 0$, w/$\alpha Re \to \infty$

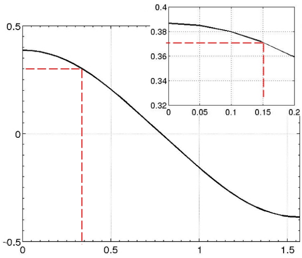
Hall & Sherwin (2010)

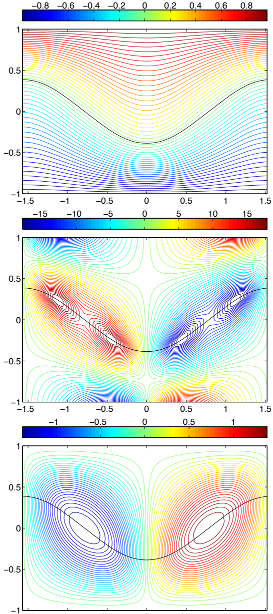
$\alpha = O(1)$



Malecha et al. (2013)

$\alpha \to 0$





Prospectus: Workshop Themes

- **Rational** asymptotic descriptions of ECS in wall flows at large *Re* possible (including **upper-branch** ECS?)

- **Ongoing Work:** ECS-based model reduction at large *Re* provides foothold for derivation of **SSP-theory in spatially-extended domains** directly from **NS eqns**

  e.g., X-modulation expected for $L_x \gg 1$ when fluctuation modes with **similar** x-wavenumbers may excited:

$$\partial_T \bar{\mathbf{v}}_{1\perp} + \partial_X \left[\bar{u}_0 \bar{\mathbf{v}}_{1\perp}\right] + \nabla_\perp \cdot \left[\bar{\mathbf{v}}_{1\perp}\bar{\mathbf{v}}_{1\perp} + \overline{\mathbf{v}'_{1\perp}\mathbf{v}'_{1\perp}}\right] \quad = \quad -\nabla_\perp \bar{p}_2 + \nabla_\perp^2 \bar{\mathbf{v}}_{1\perp}$$

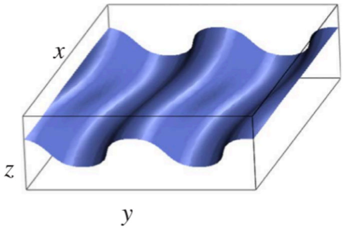
where: $\mathbf{v}'_{1\perp} = A(X,T)\hat{\mathbf{V}}_{1\perp}(y,z)e^{i(\alpha x)} + c.c.$

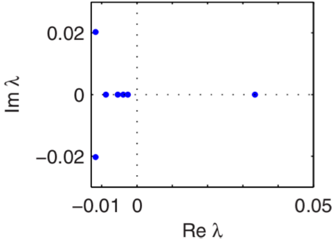
- **Future Work:** Desirable to derive **reduced PDE models** of **turbulent dynamics** in extreme parameter regimes (*a la* Julien & Knobloch for constrained convection)

  e.g., perhaps feasible to systematically derive model for interaction of superstructures with near-wall region?

Plane Couette Flow (PCF) in a $4\pi \times 2\pi \times 2$ Domain at $Re = 400$

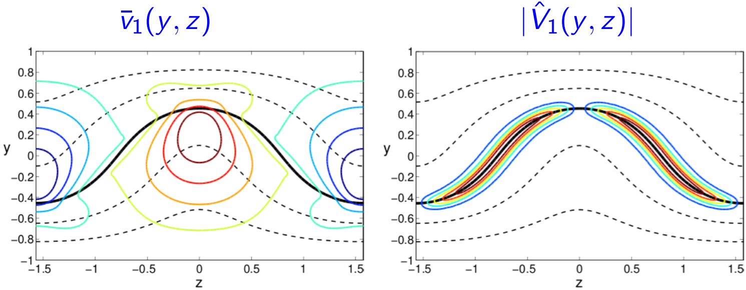
Schneider *et al.*, *Phys. Rev. E* (2008)





- **3D exact coherent states (ECS)** born in saddle-node bifurcations (arise in pairs)
- **Disconnected** from laminar base state, even as $Re \to \infty$
- **Lower-branch ECS** in PCF much studied at moderate Re and in small domains, where they possess only a small number of unstable eigen-directions
- **Upper-branch ECS** seem to capture certain statistics of uncontrolled turbulence

Lower-Branch ECS Critical Layer (CL)



Wang, Gibson & Waleffe, *PRL* (2007), $Re = 50171$

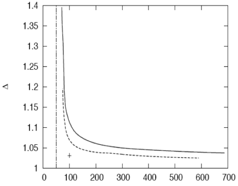
- **Fluctuations** concentrate in critical layer of thickness $O(Re^{-1/3})$
- **Mean fields** experience **jump** in x-vorticity & gradient near critical layer

Viscous Regularization of CL. II. Small-α Limit

- Derive **jump conditions** for mean x-vorticity component across CL in terms of fluctuation pressure gradient along CL $\Rightarrow$ still must solve (secondary-stability-like) **eigenvalue problem** for stability of streak motions $\bar{u}_0(y, z)$
- By further exploiting limit of long-wavelength fluctuations, i.e. $\alpha \to 0$, $\alpha Re \gg 1$, Reynolds stress **closure** can be systematically achieved $\Rightarrow$ **not** necessary to solve eigenvalue problem for fluctuation fields
- These long-wavelength states may be of interest, b/c they are the **minimum drag** states for the lower-branch ("EQ1") ECS investigated here

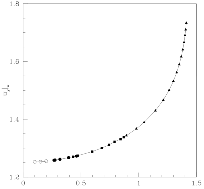
Wall Shear $\partial_y \bar{u}_0|_{y=1}$

Deguchi, Hall & Walton *JFM* (2013)



αRe

Hall & Sherwin *JFM* (2010)



α