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Reconstruction equations and the Karhunen–Loève expansion for systems with symmetry

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Abstract

We present a method for applying the Karhunen–Loève decomposition to systems with continuous symmetry. The techniques in this paper contribute to the general procedure of removing variables associated with the symmetry of a problem, and related ideas have been used in previous works both to identify coherent structures in solutions of PDEs, and to derive low-order models via Galerkin projection. The main result of this paper is to derive a simple and easily implementable set of *reconstruction equations* which close the system of ODEs produced by Galerkin projection. The geometric interpretation of the method closely parallels techniques used in geometric phases and reconstruction techniques in geometric mechanics. We apply the method to the Kuramoto–Sivashinsky equation and are able to derive accurate models of considerably lower dimension than are possible with the traditional Karhunen–Loève expansion.

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1. Introduction

KLE for symmetric systems. The Karhunen–Loève expansion (KLE), also known as the *proper orthogonal decomposition* or the method of *empirical eigenfunctions*, has been widely recognized as a useful tool both for identifying and analyzing coherent structures in turbulent fluids, and for determining low-order models for complex dynamical systems [9,20,22]. The principal idea behind the Karhunen–Loève (KL) method is that, given an ensemble of data, one can find a basis of a given dimension that spans that data optimally, in the L^2 sense.

Much of the literature on symmetry and the KL method addresses how to handle *discrete symmetries*. These discrete group considerations were first addressed by Sirovich [20], who suggested enlarging the data set by symmetry operations. These ideas were later applied by Park and Sirovich [19,21], who studied a Rayleigh–Bénard

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problem, respecting the dihedral group D_{2k}. Symmetrized data sets were further studied by Aubry et al. [2], who showed that when the data is averaged over the symmetry group, the resulting Galerkin system is equivariant with respect to the symmetry group. This is important because certain dynamical features are structurally stable only in the presence of symmetries. Berkooz and Titi [4] generalize these results to the case of general, compact Abelian groups; for discrete groups, they also suggest a means for computational savings, which was later demonstrated by Smaoui and Armbruster [23] in a study of Kolmogorov flow. The complete symmetry group for the Kolmogorov equations is the semidirect product $D_{2k} \circledS \mathrm{SO}(2)$, but the methods in [23] focus on the discrete part D_{2k}. Dellnitz et al. [6] expand further on these ideas, considering non-Abelian finite groups, and presenting a modification to the KL procedure which ensures that the Galerkin system retains precisely the same symmetry as the original system, without introducing any new symmetries.

There has also been some work on how to handle *continuous symmetries* with the KL method. It is well known that for systems with periodic or translational symmetry, the optimal basis consists of Fourier modes [20]. In systems with more general continuous symmetry groups, more complicated sets of modes can arise [4], but are nevertheless determined completely by harmonic analysis, and not from data. Such a basis normally gives no information about coherent structures in the data, and furthermore, a reduced-order model based on Fourier modes must typically retain many modes to adequately capture the dynamics. The references mentioned previously treat discrete symmetries in an efficient way, but while recognizing the importance of continuous symmetry groups and their limitations, they do not attempt to deal with these limitations.

Various methods have been developed to overcome these fundamental limitations of the KL method for systems with continuous symmetries. Such systems typically exhibit traveling structures, and several techniques have been proposed to handle them, notably those in Kirby and Armbruster [12], Armbruster et al. [1], and Glavaški [7,8]. In these works, symmetry is typically incorporated into the expansion, using for instance traveling KL modes. Traveling structures have also been considered by Cutler and Stone [5] in the context of archetypal analysis, and by Basdevant et al. [3], who present an efficient, general method for discretizing partial differential equations (PDEs) using a traveling wavelet basis.

In the traveling frame, the KL eigenfunctions are no longer forced to be Fourier modes. As a result, information about coherent structures can be obtained, and usually many fewer modes are required to accurately capture the dynamics. More generally, it is expected that if one makes use of spatial and temporal structure when applying the KL technique, then one can achieve significant computational savings. The simplest of these situations is the efficient use of symmetry methods for continuous symmetry groups, which is the subject of the present paper.

Main result of this paper. The main result of the present paper is the development of a simple and computationally efficient method for the reconstruction of traveling KL modes from their corresponding symmetry-reduced modes. This result allows one to decouple the dynamics of the mode shapes from their location and to then determine the locations by a separate integration. We demonstrate the effectiveness of the procedure using the Kuramoto–Sivashinsky equation.

Karhunen–Loève procedure. Given an ensemble of data (functions of space taken at various snapshots in time), the KL method determines a basis set of orthogonal functions of space which span the data optimally, in the L^2 sense. More precisely, if $u(x,t)$ is a function of space and time, the KL method determines functions $\varphi_n(x)$, $n = 1, 2, \ldots$, such that the projection onto the first N functions

$$\hat{u}(x,t) = \sum_{n=1}^{N} a_n(t)\varphi_n(x) \tag{1.1}$$

has a minimum error, defined by

$$E(\|u - \hat{u}\|^2). \tag{1.2}$$

Here, $E(\cdot)$ denotes time average, and $\|\cdot\|$ denotes the L^2 norm on functions of space. The functions φ_n are computed by solving the integral equation

$$\int K(x, y)\varphi(y)\,\mathrm{d}y = \lambda\varphi(x), \tag{1.3}$$

where the kernel $K(x, y) = E(u(x, t)u(y, t))$. The functions φ_n are called the *KL eigenfunctions* (also called *POD modes*, or *empirical eigenfunctions*).

If the function $u(x, t)$ is the solution to a PDE which has translational symmetry, then our method considers, instead of (1.1), the expansion

$$\hat{u}(x, t) = \sum_{n=1}^{N} a_n(t)\varphi_n(x + c(t)), \tag{1.4}$$

which is just a spatial translation of (1.1) by the amount $c(t)$. If the function u consists of a traveling structure, for instance, this expansion can be interpreted as viewing the data in the frame of reference of the traveling structure. If a Galerkin projection is to be performed on the governing PDE using the new expansion (1.4), then it is necessary to specify the evolution of the symmetry variable $c(t)$.

Reconstruction. The main contribution of this work may now be stated more precisely: we provide a simple, general method for finding *reconstruction equations* for the symmetry variable $c(t)$. The terminology "reconstruction equations" is borrowed from the geometric phase literature, as the geometric interpretation of the method closely resembles similar techniques in that literature (see, e.g. [15,16] and references therein). In our work as well as in the geometric phase literature, one of the main ideas is that one gets well-defined dynamical equations on the phase space modulo the symmetry group (these are called the *reduced equations* on the *reduced phase space*) and the problem is then how to put back into the dynamics the missing group, or phase variables. These additional equations are usually called the *reconstruction equations*.

Outline of the paper. First, in Section 2, we illustrate our method of symmetry reduction and reconstruction on a PDE that is equivariant under one-dimensional translations. The geometric interpretation of the method is then discussed in Section 3, and indicates how the method may be generalized to arbitrary continuous symmetry groups. In Section 4, we apply the method to the Kuramoto–Sivashinsky equation, which was studied in [12], and we derive low-order models which capture the dynamics over parameter ranges which are poorly modeled by the traditional methods used in [12].

2. Reduction and reconstruction: translational symmetry

First, we describe the procedure we use for determining the shift amount $c(t)$ in the expansion (1.4), essentially the position of the traveling structure. The shifting procedure we use, called *template fitting*, was introduced by Kirby and Armbruster [12] as an algorithm for preprocessing data before performing KLE. Template fitting was also used by Cutler and Stone [5] in the related context of archetypal analysis.

A similar but distinct shifting procedure, called *centering*, was introduced by Glavaški et al. [8]. This work was the first to address the *dynamics* of the projected system (1.4) (i.e., the system of ODEs obtained by Galerkin projection onto the traveling modes). In [7,8], attention was focused on PDEs of the form

$$u_t + \omega u_x = D(u), \tag{2.1}$$

where $D(\cdot)$ is a nonlinear spatial differential operator. For this case, solutions typically propagate with speed ω, so the shift variable $c(t)$ was chosen to satisfy $\dot{c}(t) = -\omega$. This is an example of a simple reconstruction equation; the

purpose of the present section is to develop *simple* reconstruction equations for more general translation-invariant PDEs than the advection equations considered in [7,8].

Either template fitting or centering may be used with the reduction techniques presented in this section, but here we focus on template fitting, which generalizes, in our view, more naturally to arbitrary symmetry groups. Centering works well for some problems, but for other problems it can lead to complicated reconstruction equations and can even fail catastrophically. We discuss centering and its limitations in Appendix A.

2.1. Template fitting

The strategy in template fitting is to shift the data so that at each time the data matches up best with a preselected template. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, so $f(x) = f(x+2\pi)$ for all x. Let $f_0(x)$ be a fixed 2π-periodic function, which will be referred to as the *template*. In [12], the shift amount c is defined to be the solution to the problem

$$\min_c \int_0^{2\pi} [f(x-c) - f_0(x)]^2 \, \mathrm{d}x, \tag{2.2}$$

where the minimization is over c in the range $0 \le c < 2\pi$. Note that solving (2.2) for c is equivalent to solving

$$\max_c \langle f(x),\, f_0(x+c)\rangle, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2[0, 2\pi]$, defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(x) g(x) \, \mathrm{d}x. \tag{2.4}$$

If c solves (2.3), then assuming differentiability, we have a critical point

$$\partial_c \langle f(x),\, f_0(x+c)\rangle = 0, \tag{2.5}$$

which is equivalent to

$$\langle f(x),\, f_0'(x+c)\rangle = 0, \quad \text{i.e.,} \quad \langle f(x-c),\, f_0'(x)\rangle = 0. \tag{2.6}$$

We shall use Eq. (2.6) to determine the shift amount c when template fitting is used. This characterization in terms of the inner product leads to a very nice geometric interpretation of template fitting, as we will see in subsequent sections.

2.2. Galerkin projection

Consider a PDE of the form

$$\partial_t u(x,t) = D(u) \tag{2.7}$$

for $0 \le x \le 2\pi$, with periodic boundary conditions and appropriate initial conditions, where $D(\cdot)$ is a nonlinear spatial differential operator that is *equivariant under spatial translations*; i.e., for each periodic function $v(\cdot)$ and each real number c,

$$D(S_c[v]) = S_c[D(v)],$$

where $S_c[v](x) = v(x+c)$ is the *shift operator* on periodic functions. Consider the function $\hat{u}$ defined by a truncated series expansion

$$\hat{u}(x,t) = \sum_{n=1}^{N} a_n(t)\varphi_n(x) + \bar{u}(x), \tag{2.8}$$

where φ_n are known orthonormal periodic functions (for us, these will be the KL eigenfunctions), and $\bar{u}$ is a known periodic function (for us this will be the mean field of the shifted solution). To find an approximate solution to Eq. (2.7), we consider

$$u(x,t) = \hat{u}(x + c(t), t), \tag{2.9}$$

where $c(t)$ is a shift amount. If $u(x,t)$ is thought of more generally as an arbitrary set of data, this procedure can be thought of as preprocessing the data, to get a shifted version of the data, $\hat{u}(x,t)$, and then performing KLE on the shifted data. Inserting the expression (2.9) into the PDE gives

$$\hat{u}_t(x+c,t) + \hat{u}_x(x+c,t)\dot{c} = D(\hat{u}(x+c,t)). \tag{2.10}$$

Note that from (2.8), we have

$$\hat{u}_t(x,t) = \sum_{n=1}^{N} \dot{a}_n(t)\varphi_n(x), \tag{2.11}$$

$$\hat{u}_x(x,t) = \sum_{n=1}^{N} a_n(t)\varphi_n'(x) + \bar{u}'(x). \tag{2.12}$$

Multiplying (2.10) by $\varphi_j(x+c)$, integrating from 0 to 2π, and using the equivariance of D gives

$$\dot{a}_j = \langle D(\hat{u}), \varphi_j\rangle - \dot{c}\langle \hat{u}_x, \varphi_j\rangle, \quad j = 1, \ldots, N. \tag{2.13}$$

This system of ordinary differential equations (ODEs) does not depend on c, but it does depend on $\dot{c}$, so to close the system we need an additional (reconstruction) equation to determine $\dot{c}(t)$.

2.3. Reconstruction equation

If we choose a template function $u_0(x)$ and define the symmetry variable $c(t)$ by template fitting, as in (2.6), then $c(t)$ satisfies

$$\langle u(x-c,t), u_0'(x)\rangle = 0. \tag{2.14}$$

Differentiating with respect to t gives

$$\langle u_t(x-c,t), u_0'(x)\rangle - \langle u_x(x-c,t), u_0'(x)\rangle\dot{c} = 0. \tag{2.15}$$

Solving for $\dot{c}$, substituting $u(x,t) = \hat{u}(x+c,t)$, and using equivariance of D, we obtain the *reconstruction equation*

$$\dot{c} = \frac{\langle D(\hat{u}), u_0'\rangle}{\langle \hat{u}_x, u_0'\rangle}. \tag{2.16}$$

This equation may be used as a closure for the system (2.13) when template fitting is used. An analogous equation may be obtained when centering is used to define the shift amount, and is discussed in Appendix A.

2.4. Summary of the method

The method consists of two main steps.

1. *Computing the reduced KL eigenfunctions*. Given an ensemble of data $u(x,t)$, one first chooses a template $u_0(x)$, and applies template fitting, forming the shifted data $\hat{u}(x,t) = u(x+c(t),t)$. Here, $c(t)$ is determined by applying Eq. (2.6) at each time t. The time average $\bar{u}(x) = E(\hat{u}(x,t))$ is then computed, and the symmetry-reduced KL eigenfunctions φ_n are found by computing the standard KLE for the zero-mean shifted data $\hat{u}(x,t) - \bar{u}(x)$.
2. *Forming the reduced model*. The dynamics of $\hat{u}(x,t) = \sum_{n=1}^{N} a_n(t)\varphi_n(x) + \bar{u}(x)$ is given by

$$\dot{a}_j = \langle D(\hat{u}), \varphi_j \rangle - \frac{\langle D(\hat{u}), u_0' \rangle}{\langle \hat{u}_x, u_0' \rangle} \langle \hat{u}_x, \varphi_j \rangle, \tag{2.17}$$

where $j = 1, \ldots, N$ (this equation is independent of c and $\dot{c}$), and then the solution is given (approximately for finite N and exactly as $N \to \infty$) by $u(x,t) = \hat{u}(x + c(t), t)$, where

$$c(t) = \int_0^t \frac{\langle D(\hat{u}(x,s)), u_0'(x) \rangle}{\langle \hat{u}_x(x,s), u_0'(x) \rangle} \, \mathrm{d}s. \tag{2.18}$$

Note that from (2.14), $\hat{u}(x,t)$ belongs to a restricted class of functions satisfying the orthogonality condition

$$\langle \hat{u}(x,t), u_0'(x) \rangle = 0. \tag{2.19}$$

The geometric meaning of this condition will be discussed in Section 3.

3. Geometric interpretation

In this section, we discuss the geometric interpretation of the above procedures, and show how the method may be generalized to arbitrary symmetry groups. Examples where more complicated symmetry groups arise include waves on a surface, where the symmetry could be the special Euclidean group SE(2) if the surface is a plane, the circle S^1 if the surface is a disk, or the special orthogonal group SO(3) if the surface is a sphere. Other interesting examples include rotating flexible structures, such as a tumbling space station, where the symmetry is again the rotation group SO(3).

3.1. Orthogonality condition

In Section 2.2, we wrote the solution u in terms of the spatial translation of a function $\hat{u}$, namely $u(x,t) = \hat{u}(x+c(t),t)$. When the translation amount $c(t)$ is defined by (2.14) then $\hat{u}(x,t)$ satisfies the orthogonality condition

$$\langle \hat{u}, u_0' \rangle = 0, \tag{3.1}$$

where $u_0(x)$ is the chosen template. Since this relation holds at any time t, this in turn implies that $\bar{u}$, the mean field of the shifted solution, is also orthogonal to u_0', and hence each of the KL eigenfunctions φ_n is also orthogonal to u_0'.

In writing the solution $u(x,t)$ as a group translation of $\hat{u}(x,t)$, and solving for the dynamics of $\hat{u}$, we have projected the solution $u(x,t)$, which lies in the set of all functions of space and time, onto a restricted set of functions $\hat{u}$ which are orthogonal to u_0'.

This procedure has the following general geometric interpretation. Consider a dynamics problem $\dot{u} = X(u)$ for a dynamical variable u, lying in a space M, and assume that there is a continuous symmetry group G that acts on

M. We will assume that M is a linear inner product space for simplicity and that the group action is linear. (The constructions hold more generally, but this is a simple case that meets our present needs.) Assume that the dynamics is given by an equivariant dynamical system on M.

In the above examples, M is the space of periodic functions, the dynamics is given by our evolution equation (i.e., X is the operator D), the inner product is the L^2 inner product and G the group of spatial translations. Equivariance just means that the equations and boundary conditions are translation invariant.

Whenever one has equivariant dynamics on M, one gets a well-defined dynamical system on the *quotient* (or *orbit*) *space* M/G which consists, in our case, of the space in which two functions related by a translation are identified. When M is an inner product space and the group action is by isometries, there is a natural way to identify, at least locally in function space, the quotient space with a subspace of M; namely we pick a point $u_0 \in M$ and look at the affine space through the point u_0 orthogonal to the group orbit through that point. [1] We call this affine space a *slice* and denote it by S_{u_0}.

In our case, the orthogonality condition defining the space S_{u_0} is exactly the condition in Eq. (3.1). Indeed, the tangent space to the group orbit is the one-dimensional space (since $G = \mathbb{R}$ is one-dimensional) given by differentiating the translation of the function u_0 by an amount c with respect to c at the identity, $c = 0$. This is, of course, just the function u_0'. The affine space S_{u_0} is then defined as

$$S_{u_0} = \{u_0 + \hat{u} | \langle \hat{u}, u_0' \rangle = 0\} \tag{3.2}$$

or, equivalently,

$$S_{u_0} = \{\hat{u} | \langle \hat{u}, u_0' \rangle = 0\}, \tag{3.3}$$

since $\langle u_0, u_0' \rangle = 0$ (this identity holds for all periodic functions u_0).

In the more general theory, assuming that the point u_0 has no isotropy (in our case this means that the function u_0 is not symmetric with respect to any nontrivial translations by amounts strictly between 0 and 2π), the map that identifies an element of S_{u_0} with its equivalence class in M/G is a local diffeomorphism. One can also identify (modulo points with isotropy) M, at least locally, with the product space $M/G \times G$, i.e., with $S_{u_0} \times G$. The identification takes an element $(r, g) \in S_{u_0} \times G$ and maps it to the element of M given by the action of g on r.

One wants now to reconstruct the dynamics on M from the dynamics on M/G and the reconstruction equation provides the missing dynamics for the group elements. This is exactly what we are doing here.

In the reconstruction and geometric phase literature in mechanics, one often exploits an inner product structure as well via a *connection*, and in that theory the reconstruction equations have the same flavor as those we have obtained here. In Section 3.2, we give a reconstruction equation in a simplified setting appropriate to our needs and give the reconstruction equation using connections in Appendix B.

3.2. Reconstruction equation from slices

The general procedure is indicated in Fig. 1. Consider a dynamical system which evolves in a space M, and which admits a continuous symmetry group G. In particular, for $u(t) \in M$, $u(t)$ satisfies

$$\dot{u} = X(u), \tag{3.4}$$

where the differential operator X is equivariant under the action of G. (This corresponds to the operator D in preceding sections.)

[1] See, e.g. [17] for an elementary discussion of these group theoretic concepts.

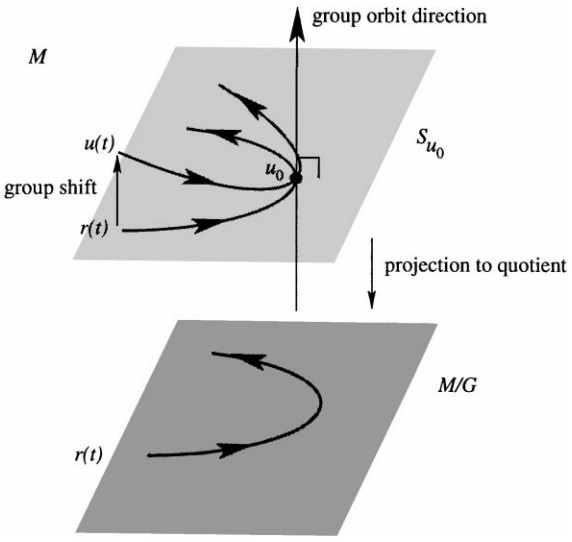

Fig. 1. The geometry of the reconstruction equation.

To derive reconstruction equations, we begin by generalizing the orthogonality condition (3.1). The above discussion of the geometric viewpoint suggests a natural way of generalizing this condition. We begin by choosing a point $u_0 \in M$ (the template), and constructing the tangent space to the group orbit, defined by

$$T_{u_0}\mathrm{Orb}(u_0) = \{\xi_M(u_0) | \xi \in \mathfrak{g}\}, \tag{3.5}$$

where $\mathfrak{g}$ is the Lie algebra of G, and $\xi_M : M \to TM$ (a vector field on M) denotes the infinitesimal generator of the action corresponding to ξ. Then, the slice S_{u_0} consists of all functions r (corresponding to $\hat{u}$ previously), which are orthogonal to this tangent space, so that

$$S_{u_0} = \{u_0 + r | \langle r, \xi_M(u_0)\rangle = 0 \quad \forall \xi \in \mathfrak{g}\}. \tag{3.6}$$

Now we identify the quotient space (locally in the space M) with the slice, as explained above. In other words, we define $r(t) \in S_{u_0}$ by translating $u(t)$ until it hits the slice. The general theory (see also Appendix B) guarantees that the function r inherits well-defined dynamics. We verified this directly for our example in Section 3.1. The resulting quotient dynamics will be denoted

$$\dot{r}(t) = X_{S_{u_0}}(r) \tag{3.7}$$

(in the setting of Appendix B, this is denoted as $[X]$). We now start with a solution of the quotient dynamics $r(t)$ and attempt to reconstruct the solution $u(t)$.

To do this, we seek a group element $g(t)$ such that $u(t) = g(t)r(t)$ (the group action is denoted by concatenation) satisfies the given Eq. (3.4). To derive the equation for $g(t)$, substitute $u(t) = g(t)r(t)$ into $\dot{u}(t) = X(u(t))$ to give an equation in $\dot{r}$ and $\dot{g}$ which we denote

$$\dot{g}r + g\dot{r} = X(gr). \tag{3.8}$$

(Appendix B gives a more general formula.) Using (3.7) and equivariance of X (i.e., $X(gr) = gX(r)$), (3.8) is equivalent to

$$g^{-1}\dot{g} \cdot r + X_{S_{u_0}}(r) = X(r). \tag{3.9}$$

As shown in Appendix B, the precise way to interpret this equation is as follows. Let $\xi(t) = g(t)^{-1}\dot{g}(t)$ (left translation of $\dot{g}(t)$ to the identity), which is a curve in the Lie algebra $\mathfrak{g}$. The first term of the left-hand side of (3.9) is exactly $(\xi(t))_M(r(t))$.

Consider now the orthogonal projection map $\mathbb{P} : M \to S_{u_0}$. The orthogonal projection to the complement, namely $\mathrm{Id} - \mathbb{P}$, takes a vector v in M and produces a vector tangent to the group orbit through u_0. We now apply $\mathrm{Id} - \mathbb{P}$ to Eq. (3.9). Since, by construction, $(\mathrm{Id} - \mathbb{P})X_{S_{u_0}}(r(t)) = 0$, we get

$$(\mathrm{Id} - \mathbb{P})(\xi(t) \cdot r(t)) = (\mathrm{Id} - \mathbb{P})X(r(t)), \tag{3.10}$$

which may be regarded as an algebraic equation to be solved for ξ. This gives the *reconstruction equation*, denoted by

$$\xi = \xi(r), \tag{3.11}$$

which then yields a differential equation for $\dot{g}$.

The equation for the dynamics of r itself is then obtained from (3.9)

$$\dot{r} = X_{S_{u_0}}(r) = X(r) - \xi(r) \cdot r. \tag{3.12}$$

Special case: one-dimensional translational symmetry. We now show that the template reconstruction equation (2.16) is indeed a special case of Eq. (3.10), when G is the group of one-dimensional translations or rotations, so the Lie algebra is simply $\mathfrak{g} = \mathbb{R}$.

In this case, the group actions and generators are given by

$$(gu)(x,t) = u(x+g,t), \qquad (\xi \cdot u)(x,t) = \xi u_x(x,t),$$

where $g \in G$ and $\xi \in \mathfrak{g}$. The slice S_{u_0} is defined by Eq. (3.3), and the orthogonal projection to the complement of the slice is given by

$$(\mathrm{Id} - \mathbb{P})(v) = \frac{\langle v, u_0' \rangle}{\langle u_0', u_0' \rangle} u_0'.$$

In our case, $\xi = \dot{c}$, so (3.10) becomes

$$\dot{c}\frac{\langle \hat{u}_x, u_0' \rangle}{\langle u_0', u_0' \rangle} u_0' = \frac{\langle D(\hat{u}), u_0' \rangle}{\langle u_0', u_0' \rangle} u_0',$$

which, after taking an inner product with u_0', agrees with (2.16). Substituting this equation into (3.12) gives, as before, the dynamics of $\hat{u}$ itself.

4. Application: the Kuramoto–Sivashinsky equation

We now apply the symmetry reduction procedure described above to a sample problem, the Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0 \tag{4.1}$$

for $0 \le x \le 2\pi$, with periodic boundary conditions. Several versions of the KS equation have been studied; perhaps the most common form is

$$v_\tau + 4v_{xxxx} + \alpha(v_{xx} + \tfrac{1}{2}v_x^2) = 0, \tag{4.2}$$

which is equivalent to (4.1) with

$$u = v_x, \qquad t = \alpha\tau, \qquad \nu = 4/\alpha. \tag{4.3}$$

The dynamics of (4.2), for a wide range of the parameter α, have been extensively investigated by Hyman et al. [10], and traditional KLE and Galerkin projection were applied to this form of the equation by Kirby and Armbruster [12]. One reason the form (4.2) is often preferred in the dynamics literature is that it has greater symmetry, since (4.2) is O(2)-equivariant, while (4.1) is only SO(2)-equivariant. However, despite the loss of symmetry, the form (4.1) has several nice features. First, it bears closer resemblance to other model problems of fluid dynamics, such as Burger's equation. In addition, the spatial average

$$u_m(t) := \int_0^{2\pi} u(x,t)\,\mathrm{d}x$$

remains constant in time, while for the form (4.2) the corresponding mean quantity is not constant, and simulations of this equation typically add a correction term to keep the mean value from growing unbounded (see, e.g. [10]).

Here, we begin by computing an accurate numerical solution to (4.1) for several different values of the parameter ν. Details of the computation are included in Section 4.1 along with the Galerkin ODEs and reconstruction equations for the KS equation. We then apply template fitting to the data, compute the KL eigenfunctions from the shifted data, and finally solve the low-order system, and compare solutions of the reduced system to those of the full system.

4.1. Numerical details

We first compute a highly accurate solution to (4.1) using a 20-mode (complex) Fourier–Galerkin representation, and using a Crank–Nicholson scheme to advance the linear terms and second-order Adams–Bashforth to advance the nonlinear terms. Because of the sensitive dependence on initial conditions, all computations were performed in double precision, and through a careful study of convergence in space and time, we determined that 20 modes and a timestep of 10^{-4} were sufficient to accurately compute a solution for the parameter values we investigated.

We then shift the data, using template fitting, and in our examples we take the template to be the first snapshot (i.e., $u_0(x) = u(x,t_0)$). We subsequently subtract the mean field of the shifted data

$$\bar{u}(x) = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \hat{u}(x,t)\,\mathrm{d}t, \tag{4.4}$$

where t_0 and t_1 are the times of the first and last snapshots used, and then compute the standard KL eigenfunctions $\varphi_n(x)$ for the shifted, zero-mean data. Because the computational data is given in terms of Fourier modes, the KL eigenfunctions are also computed in terms of their Fourier coefficients, using the method of snapshots (see, e.g. [20]).

Once we have the spatial modes $\varphi_n(x)$, we may apply the Galerkin projection discussed in Section 2.2. Taking

$$D(u) = -uu_x - u_{xx} - \nu u_{xxxx}, \tag{4.5}$$

and writing u as the KLE

$$u(x,t) = \sum_{n=1}^{N} a_n(t)\varphi_n(x + c(t)) + \bar{u}(x + c(t)), \tag{4.6}$$

the Galerkin projection of Section 2.2 yields the system of ODEs

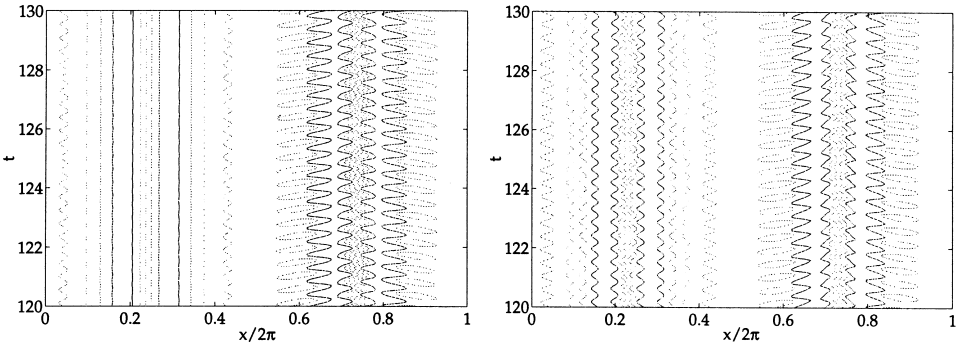

Fig. 2. Contour plot of solution of full simulation for $\alpha = 84.25$; same solution after template fitting.

$$\dot{a}_k = -\sum_{m,n=1}^{N} b_{kmn} a_m a_n - \sum_{n=1}^{N} c_{kn} a_n - d_k - \sum_{n=1}^{N} e_{kn} a_n \dot{c} - f_k \dot{c} \tag{4.7}$$

for $k = 1, \ldots, N$, where

$$b_{kmn} = \langle \varphi_n \varphi_m', \varphi_k \rangle, \qquad c_{kn} = \langle \bar{u}\varphi_n' + \bar{u}'\varphi_n + \varphi_n'' + \nu\varphi_n'''', \varphi_k \rangle, \qquad d_k = \langle \bar{u}\bar{u}' + \bar{u}'' + \nu\bar{u}'''', \varphi_k \rangle,$$
$$e_{kn} = \langle \varphi_n', \varphi_k \rangle, \qquad f_k = \langle \bar{u}', \varphi_k \rangle,$$

are constants which may be computed before solving (4.7). The derivatives in these coefficients may be computed exactly (without finite differencing), since the KL modes, mean field, and template are all known in terms of their Fourier coefficients. To close this system, we use the reconstruction equation (2.16), which takes the form

$$\dot{c} = -\frac{\sum_{m,n=1}^{N} p_{mn} a_m a_n + \sum_{n=1}^{N} q_n a_n + r}{\sum_{n=1}^{N} s_n a_n + t}, \tag{4.8}$$

where

$$p_{mn} = \langle \varphi_n \varphi_m', u_0' \rangle, \qquad q_n = \langle \bar{u}\varphi_n' + \bar{u}'\varphi_n + \varphi_n'' + \nu\varphi_n'''', u_0' \rangle, \qquad r = \langle \bar{u}\bar{u}' + \bar{u}'' + \nu\bar{u}'''', u_0' \rangle,$$
$$s_n = \langle \varphi_n', u_0' \rangle, \qquad t = \langle \bar{u}', u_0' \rangle.$$

We solve this reduced system using a fourth-to fifth-order variable-timestep Runge–Kutta method, with an error tolerance of 10^{-6}, and compare the solution of the reduced system to the solution of the full system, obtained from the 20-complex-mode Fourier–Galerkin procedure.

4.2. Full simulations and template fitting

We study numerical solutions of the KS equation for two different values of the parameter $\alpha = 84.25$ and $\alpha = 87$, where $\nu = 4/\alpha$ is the parameter in (4.1). This regime has been studied extensively in [10], and low-order models were derived in [12]. For $72 < \alpha < 89$, there exists a strange [2] fixed point which is globally attracting. Solutions in the vicinity of the fixed point consist of beating waves, which are stationary for $\alpha < 86$ and traveling for $\alpha > 86$.

Fig. 2 shows the contour plot of the beating wave for $\alpha = 84.25$. The contour levels for all plots are between -10 and 10, equally spaced at intervals of 5.0. Also shown is the solution after template fitting has been performed

[2] As coined in [10], this fixed point is called "strange" because it is not a cellular state, and has a broad Fourier spectrum. This should not be confused with the notion of a strange (i.e., chaotic) attractor encountered in nonlinear ODEs.

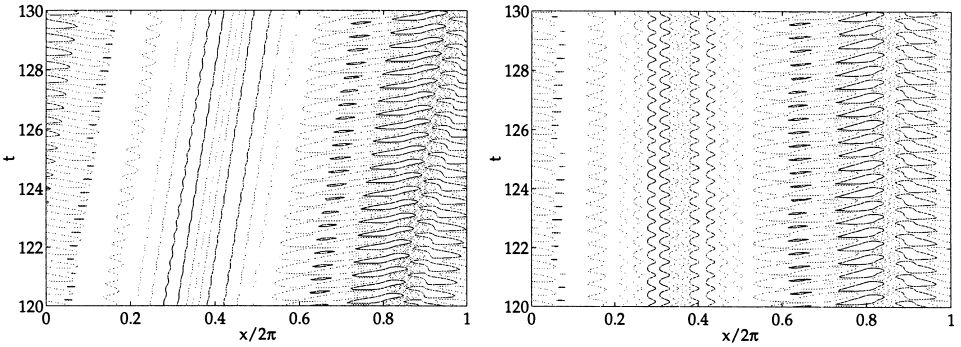

Fig. 3. Contour plot of solution of full simulation for $\alpha = 87$; same solution after template fitting.

(i.e., the left plot shows $u(x,t)$, and the right plot shows $\hat{u}(x,t) = u(x-c,t)$). As stated earlier, in all cases the template u_0 was chosen to be the first snapshot.

Fig. 3 shows the solution before and after template fitting for $\alpha = 87$. Note that the beating wave is now traveling in space, and the template fitting removes this translation. The initial condition for all runs is the same as that used in [12]:

$$u(x,0) = -\sin(x) + 2\cos(2x) + 3\cos(3x) - 4\sin(4x),$$

and the transient solution is computed until $t = 120$, by which time a relative equilibrium has been reached. Solutions were computed from $t = 120$ to $t = 140$, but for clarity, Figs. 2 and 3 show the solution only to $t = 130$.

4.3. Reduced-order simulations

KL modes were determined from the above data by taking 400 snapshots between $t = 120$ and $t = 140$. The reduced equations (4.7) and (4.8) were then computed, and solved keeping various numbers of modes. We found that keeping three modes was sufficient to capture the qualitative behavior for either parameter value. When four or more modes are used, the solution of the reduced system is virtually indistinguishable from the solution of the full simulation.

Recall that without the symmetry reduction methods discussed here, the optimal modes are Fourier modes. If Fourier modes are used, eight complex modes (or 16 real modes) are required to capture qualitatively correct dynamics, for either value of α shown here. (When fewer than eight Fourier modes are used, either the oscillations die out completely, or the solution blows up.) Figs. 4 and 5 show the 3-mode solution of the reduced equations along with the solution from the 8-mode (complex) Fourier–Galerkin simulation.

We remark that in Kirby and Armbruster [12], a 3-mode model was obtained for a regime where solutions do not travel ($\alpha = 72$), but no low-order models were attempted for $\alpha = 87$, where the beating wave is traveling. Also, in our simulations of the reduced system we experienced none of the stability problems reported in [12].

5. Discussion

We have presented a technique for combining symmetry reduction techniques with the KL method. The main result has been to derive reconstruction equations which specify the evolution of the symmetry variable $c(t)$. For the simple case of one-dimensional translational symmetry, the reconstruction equation is particularly simple, and we have generalized our approach to an abstract setting, which applies to a large class of continuous symmetry groups.

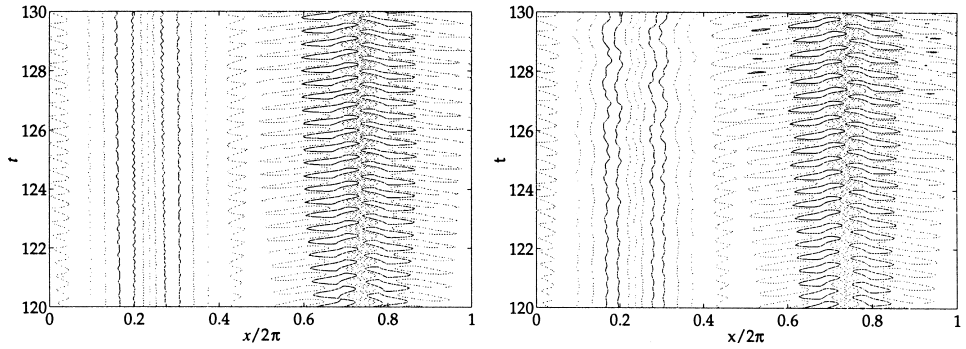

Fig. 4. Contour plot of solution of 3-mode KL–Galerkin system for $\alpha = 84.25$; solution of corresponding 8-mode (complex) Fourier–Galerkin system.

We applied the method to the KS equation, and were able to derive models of much lower order than were previously possible. Our method is particularly effective when solutions are traveling waves, in which case the standard KLE gives Fourier modes. For the example shown, the standard method requires at least eight complex Fourier modes (16 degrees of freedom) to capture qualitatively correct dynamics, while our method requires only three modes.

In recent years, there have been significant developments in incorporating symmetry into the KLE, mostly focusing on discrete symmetry groups (e.g. [2,23]). The method presented in this paper for continuous groups nicely complements these methods for discrete groups, and in fact may permit them to be used even more effectively on problems with both continuous and discrete symmetry.

Another structure that is important to take into account in some situations is the mechanical structure. For elastic systems this structure is used in Lall et al. [13]; similar things should also be of interest in fluid mechanics. It would be natural to extend the reduction procedures here to similar ones for mechanical systems that exploit both reduction theory for mechanical systems and variational symplectic integration methods such as the Newmark algorithm (see [11,18] for the current state of affairs in these subjects and for further references).

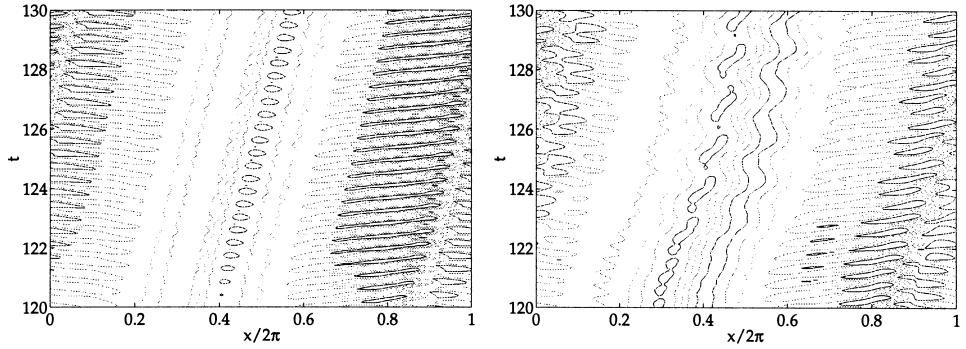

Fig. 5. Contour plot of solution of 3-mode KL–Galerkin system for $\alpha = 87$; solution of corresponding 8-mode (complex) Fourier–Galerkin system.

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Appendix A. Centering: an alternative shifting procedure

This appendix discusses *centering*, a shifting procedure which may be used in place of template fitting. The method was introduced in [7,8], where it was applied to models of rotating stall cells in compressors.

We define a *center* of a 2π-periodic function $f(x)$ to be a value of $c \in [0, 2\pi)$ that satisfies

$$\int_0^\pi [f(x-c)]^2 \,\mathrm{d}x = \int_\pi^{2\pi} [f(x-c)]^2 \,\mathrm{d}x. \tag{A.1}$$

This definition of the center does *not* correspond to the definition of center in [7,8], but corresponds rather to the final shift value reached after applying the iteration described in [7,8].[3] If c satisfies (A.1), we say that the new function $f_c(x) = f(x-c)$ is *centered*.

A.1. Reconstruction equation: centering

Consider the translationally invariant PDE $\dot{u} = D(u)$, as before, and now define $c(t)$ such that

$$\int_0^\pi [u(x-c,t)]^2 \,\mathrm{d}x = \int_\pi^{2\pi} [u(x-c,t)]^2 \,\mathrm{d}x \tag{A.2}$$

for all t. Differentiate this relation with respect to t, to give

$$\begin{aligned}&\int_0^\pi 2u(x-c,t)(u_t(x-c,t) - u_x(x-c,t)\dot{c}) \,\mathrm{d}x \\ &\quad = \int_\pi^{2\pi} 2u(x-c,t)(u_t(x-c,t) - u_x(x-c,t)\dot{c}) \,\mathrm{d}x.\end{aligned} \tag{A.3}$$

Letting $\hat{u}(x,t) = u(x-c,t)$ and noting that $u_x(x,t) = \hat{u}_x(x+c,t)$ and $u_t(x,t) = D(u(x,t))$, this becomes

$$\int_0^\pi \hat{u} \cdot (D(\hat{u}) - \hat{u}_x\dot{c}) \,\mathrm{d}x = \int_\pi^{2\pi} \hat{u} \cdot (D(\hat{u}) - \hat{u}_x\dot{c}) \,\mathrm{d}x, \tag{A.4}$$

where we have used equivariance of D. Solving for $\dot{c}$, we have

$$\dot{c} = \frac{\int_0^\pi \hat{u}D(\hat{u}) \,\mathrm{d}x - \int_\pi^{2\pi} \hat{u}D(\hat{u}) \,\mathrm{d}x}{\int_0^\pi \hat{u}\hat{u}_x \,\mathrm{d}x - \int_\pi^{2\pi} \hat{u}\hat{u}_x \,\mathrm{d}x}. \tag{A.5}$$

[3] The center c defined in [7,8] satisfies $\int_0^c |f|^2 \,\mathrm{d}x = \int_c^{2\pi} |f|^2 \,\mathrm{d}x$. One then sets up an iteration procedure to produce a *shift* d which corresponds to our c.

This equation, which we refer to as the *centering reconstruction equation*, may be used as a closure for the system (2.13), when $c(t)$ is determined by centering (i.e., defined by (A.2)). Defining the *centering bilinear functional* $\langle\langle\cdot,\cdot\rangle\rangle_c : L^2 \times L^2 \to \mathbb{R}$ by

$$\langle\langle f, g\rangle\rangle_c = \int_0^\pi f(x)g(x)\,\mathrm{d}x - \int_\pi^{2\pi} f(x)g(x)\,\mathrm{d}x, \tag{A.6}$$

we may rewrite (A.5) in the concise form as

$$\dot{c} = \frac{\langle\langle D(\hat{u}), \hat{u}\rangle\rangle_c}{\langle\langle \hat{u}_x, \hat{u}\rangle\rangle_c}, \tag{A.7}$$

which closely resembles the form of the template fitting reconstruction equation (2.16).

Note that when $u(x,t) = \hat{u}(x + c, t)$ and $c(t)$ is determined by centering, then $\hat{u}(x,t)$ belongs to a restricted class of "centered" functions satisfying

$$\langle\langle \hat{u}, \hat{u}\rangle\rangle_c = 0, \quad \text{i.e.,} \quad \int_0^\pi [\hat{u}(x,t)]^2\,\mathrm{d}x = \int_\pi^{2\pi} [\hat{u}(x,t)]^2\,\mathrm{d}x \tag{A.8}$$

for all t. This space of centered functions is the analog of the slice S_{u_0} defined in Section 3.1 for the template fitting reduction procedure.

A.2. Limitations

Centering works well for many problems, but for certain problems the method can fail catastrophically. The problem is that the derivation of the reconstruction equation in Section A.1 requires that the shift amount $c(t)$ is differentiable. When centering is used to define $c(t)$, this amount can change discontinuously, even when the solution $u(x,t)$ varies smoothly in time. We illustrate this process with the following example.

Consider the solution of the KS equation, for $\alpha = 84.25$, as discussed in Section 4.2. Part of the solution is plotted in Fig. 6, both before and after centering is applied (i.e., the left plot shows $u(x,t)$, and the right plot shows $\hat{u}(x,t) = u(x + c, t)$, where $c(t)$ is given by (A.2)). As the beating wave oscillates, the center location jumps back and forth discontinuously. This jumping is an inherent flaw in the method, and not dependent on the algorithm which finds the (possibly nonunique) location of the center, as we now show.

For a given function $u(x,t)$, possible center locations (i.e., solutions of (A.2)) correspond to zeros of the function

$$F_t(c) = \langle\langle u(x - c, t), u(x - c, t)\rangle\rangle_c.$$

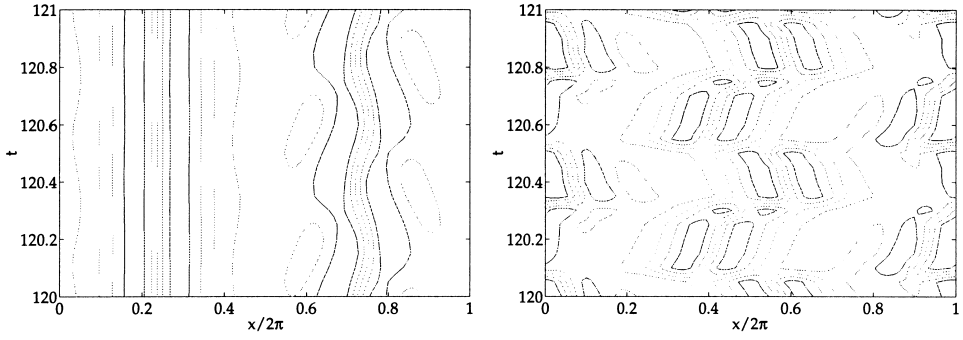

Fig. 6. Contour plot of solution of full simulation for $\alpha = 84.25$; same solution after centering — catastrophic failure.

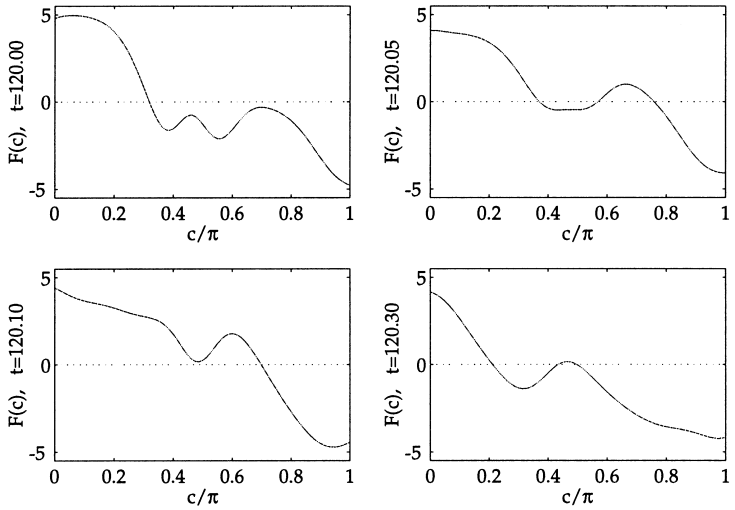

Fig. 7. Plot of $F_t(c)$ at four different times ($t = 120.00$, 120.05, 120.10, and 120.30). Note how zeros of $F_t(c)$ must change discontinuously as t varies.

Note that $F_t(c+\pi) = -F_t(c)$ for all c, so we need only search for zeros in the range $0 \leq c < \pi$. Fig. 7 shows the function $F_t(c)$ at several different times, for the solution u plotted in Fig. 6.

There is a unique solution of $F_t(c) = 0$ for c at the initial time. At the next time shown in Fig. 7, multiple solutions arise, but we may still follow the original solution. At the third time, the original solution disappears, and we are forced to jump discontinuously to a new (unique) root. At the final time shown, multiple roots arise once again, and the process repeats itself.

It is possible that this difficulty may arise with template fitting as well, in fact, a straightforward application of the implicit function theorem shows that this jumping is possible with template fitting at points $v \in S_{u_0}$, where the group orbit $\mathrm{Orb}(v)$ is tangent to the slice S_{u_0}. (In the one-dimensional example, for instance, this is when the denominator in the reconstruction equation (2.16) becomes zero.) Locally, in a neighborhood of the template u_0, this situation is guaranteed not to occur, as long as the action is locally free. However, if the dynamics carry the solution far from the template, this difficulty may arise. In this case, one might choose a new template near the new dynamics, and treat the different slices as local coordinate charts for the quotient space M/G.

Appendix B. General reconstruction equation

In this appendix, we give an alternative abstract reconstruction equation in the setting of general dynamical systems with symmetry. When one has the additional structure of a mechanical system with conservation laws one can refine this procedure by taking the conservation law into account (see, e.g. [15,18] for this theory).

B.1. Equivariant dynamical systems

The general theory starts with a manifold M with the (left) action of a Lie group G. The Lie algebra of G is denoted $\mathfrak{g}$. We denote the action of a group element $g \in G$ on a point $u \in M$ by $\Phi_g(u) = g \cdot u$. We consider an equivariant dynamical system $\dot{u} = X(u)$ on M. We assume that the action of G is free and proper so that the quotient

or orbit space M/G is a smooth manifold. We denote the projection to the quotient by $\pi : M \to M/G; u \mapsto [u]$, where $[u] = \{g \cdot u | g \in G\}$ denotes the equivalence class of u.

Choosing a representative u of the class $[u]$, the tangent space to M/G at a point $[u]$ is isomorphic to the quotient space $T_u M/(\xi \cdot u)$. Here, $\xi \cdot u = \{\xi_M(u) | \xi \in \mathfrak{g}\}$ is the tangent space to the group orbit through the point u, where $\xi_M(u)$ denotes the infinitesimal generator of the group action associated with the Lie algebra element $\xi \in \mathfrak{g}$. The isomorphism mentioned is induced by the tangent to the projection map: $T\pi : TM \to T(M/G)$.

Because the vector field X is equivariant, it induces a vector field $[X]$ on the quotient space with the property that $[X]([u]) = T_u\pi \cdot X(u)$ for all $u \in M$. The flow φ_t of $[X]$ is related to the flow F_t of X by $\pi \circ F_t = \varphi_t \circ \pi$.

B.2. Reconstruction

The reconstruction problem is the following. Given an integral curve $[u](t) = \varphi_t([u](0))$ of the quotient dynamics and a point u_0 such that $[u_0] = [u](0)$, determine the solution $u(t)$ with initial condition u_0.

Connections. To carry out this we need some additional structure, namely that of a connection. Recall that a (principal) connection is a Lie algebra valued one form $\mathcal{A} : TM \to \mathfrak{g}$ with the following properties:

1. $\mathcal{A}(\xi_M(u)) = \xi$ for all $\xi \in \mathfrak{g}$ and $u \in M$.
2. $\mathcal{A}$ is equivariant (with respect to the given action on M and the adjoint action on $\mathfrak{g}$), i.e., for a tangent vector $v_u \in T_u M$,

$$\mathcal{A}(g \cdot v_u) = \mathrm{Ad}_g(\mathcal{A}(v_u)),$$

   where $g \cdot v_u$ denotes the tangent action of G on TM.
3. The horizontal space $\mathrm{Hor}_u = \ker \mathcal{A}|T_u M$ is a complement to the vertical space $\xi \cdot u$.

Given a connection, one has a horizontal and vertical decomposition of any vector $v_u \in T_u M$ as follows:

$$v_u = \mathrm{Ver}_u(v_u) + \mathrm{Hor}_u(v_u),$$

where $\mathrm{Ver}_u(v_u) = (\mathcal{A}(v_u))_M(u)$ and $\mathrm{Hor}_u(v_u) = v_u - \mathrm{Ver}_u(v_u)$.

If one has an equivariant distribution of horizontal spaces, then these properties uniquely determine a connection.

For example, if we can write $M = S \times G$ and the group action is by left translation on the second factor alone, then a connection is given by declaring the first factor to be horizontal and the second factor to be vertical.

The reconstruction equation. The reconstruction equation is based on a general formula for the derivative of a curve of the form $u(t) = g(t) \cdot z(t)$, where $g(t)$ is a curve in G and $z(t)$ is a curve in M. This formula is the following:

$$\dot{u}(t) = (\mathrm{Ad}_{g(t)}\xi(t))_M(u(t)) + g(t) \cdot \dot{z}(t) = g(t) \cdot [(\xi(t))_M(z(t)) + \dot{z}(t)],$$

where $\xi(t) = g(t)^{-1} \cdot \dot{g}(t)$ is a curve in the Lie algebra. This formula is proved in, e.g. [17].

Given the integral curve $[u](t)$ of $[X]$ in M/G, we choose a convenient curve $\tilde{u}(t)$ with the property that $[\tilde{u}(t)] = [u](t)$ and $\tilde{u}(0) = u_0$. We then write our solution in the form $u(t) = g(t)\tilde{u}(t)$. By the preceding display, we can write

$$\dot{u}(t) = g(t) \cdot [(\xi(t))_M(\tilde{u}(t)) + \dot{\tilde{u}}(t)].$$

We now use the fact that $u(t)$ should be an integral curve of X and the fact that X is equivariant to get

$$X(\tilde{u}(t)) = (\xi(t))_M(\tilde{u}(t)) + \dot{\tilde{u}}(t).$$

Apply the connection to both sides of this equation

$$\mathcal{A}(X(\tilde{u}(t))) = \xi(t) + \mathcal{A}(\dot{\tilde{u}}(t)).$$

Solving for ξ gives the desired *reconstruction equation* :

$$g(t)^{-1} \cdot \dot{g}(t) = \mathcal{A}(X(\tilde{u}(t) - \dot{\tilde{u}}(t))), \tag{B.1}$$

which we regard as a differential equation for the unknown curve $g(t)$ to be solved with the initial condition $g(0) = \mathrm{Id}$. With this solution, the desired integral curve of X is given by $u(t) = g(t) \cdot \tilde{u}(t)$.

The technique we used in the text to derive the reconstruction equation on slices is *not literally* a special case of the geometric version given here, but it proceeds in the same spirit. To illuminate the similarities and distinctions between the two approaches, we now write the reconstruction equation (B.1), where G is the group of one-dimensional translations.

Special case: one-dimensional translational symmetry. We begin by constructing the principal connection $\mathcal{A}$, assuming the additional structure of a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on M. The procedure we follow is a standard procedure for constructing the *mechanical connection* as described in [14].

First, for each $u \in M$, we define the *locked inertia tensor* $I(u) : \mathfrak{g} \to \mathfrak{g}^*$ by

$$\langle I(u)\xi, \eta \rangle = \langle\langle \xi_M(u), \eta_M(u) \rangle\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing. In our case, $\xi_M(u) = \xi u'$ and the natural pairing is just scalar multiplication, so $I(u) = \langle\langle u', u' \rangle\rangle$.

Next, we define the *momentum map* $J : TM \to \mathfrak{g}^*$ by

$$\langle J(v_u), \xi \rangle = \langle\langle v_u, \xi_M(u) \rangle\rangle,$$

which in our case is just $J(v_u) = \langle\langle v_u, u' \rangle\rangle$. Finally, the connection $\mathcal{A} : TQ \to \mathfrak{g}$ is given by

$$\mathcal{A}(v_u) = I(u)^{-1} \cdot J(v_u),$$

which for translational symmetry is simply

$$\mathcal{A}(v_u) = \frac{\langle\langle v_u, u' \rangle\rangle}{\langle\langle u', u' \rangle\rangle}. \tag{B.2}$$

It is simple to verify that $\mathcal{A}$ satisfies the three properties of a principal connection.

For translational symmetry, $g(t)^{-1} \cdot \dot{g}(t) = \dot{g}(t)$, and so the reconstruction equation (B.1) becomes

$$\dot{g}(t) = \frac{\langle\langle X(\tilde{u}) - \dot{\tilde{u}}, \tilde{u}' \rangle\rangle}{\langle\langle \tilde{u}', \tilde{u}' \rangle\rangle}. \tag{B.3}$$

A natural way to specify the (arbitrary) choice of $\tilde{u}$ is that it be horizontal (i.e., $\dot{\tilde{u}}(t) \in \mathrm{Hor}_{\tilde{u}(t)}$ for all t). Then Eq. (B.3) becomes

$$\dot{g}(t) = \frac{\langle\langle X(\tilde{u}), \tilde{u}' \rangle\rangle}{\langle\langle \tilde{u}', \tilde{u}' \rangle\rangle}, \tag{B.4}$$

which is identical to the reconstruction equation on slices (2.16), if the (now time-varying) template u_0 is the solution itself $\hat{u}(t)$.

Application to the KL method. As described in the summary in Section 2.4, there are two distinct steps in applying these symmetry methods to the KL procedure. The first step involves computing the KL eigenfunctions

for symmetry-reduced space, by first shifting data by the group action, and then performing the standard KLE. The second step involves constructing the reduced-order model, and this is where the reconstruction equation is needed.

The reconstruction procedures given in this appendix address only the second part of the KL method. They allow one to construct a Galerkin model of a PDE, provided the KL eigenfunctions are already specified. They do not, however, indicate how to construct the KL eigenfunctions in the symmetry-reduced space. To accomplish this, one needs a shifting procedure, such as template fitting, or its generalization, shifting onto slices, described in Section 3.2.

References

[1] D. Armbruster, E. Stone, R. Heiland, Towards analyzing the dynamics of flames, Fields Inst. Commun. 5 (1996) 1–17.
[2] N. Aubry, W.-Y. Lian, E.S. Titi, Preserving symmetries in the proper orthogonal decomposition, SIAM J. Sci. Comput. 14 (1993) 483–505.
[3] C. Basdevant, M. Holschneider, V. Perrier, Méthode des ondelettes mobiles, C. R. Acad. Sci. Paris Ser. I 310 (1990) 647–652.
[4] G. Berkooz, E.S. Titi, Galerkin projections and the proper orthogonal decomposition for equivariant equations, Phys. Lett. A 174 (1993) 94–102.
[5] A. Cutler, E. Stone, Moving archetypes, Physica D 107 (1997) 1–16.
[6] M. Dellnitz, M. Golubitsky, M. Nicol, Symmetry of attractors and the Karhunen–Loève decomposition, in: Trends and Perspectives in Applied Mathematics, Applied Mathematical Sciences, Vol. 100, Springer, Berlin, 1994, pp. 73–108.
[7] S. Glavaški, Robust system analysis and nonlinear system model reduction, Ph.D. Thesis, California Institute of Technology, May 1998.
[8] S. Glavaški, J.E. Marsden, R.M. Murray, Model reduction, centering, and the Karhunen–Loève expansion, Proc. CDC 37 (1998) 2071–2076.
[9] P. Holmes, J.L. Lumley, G. Berkooz, Turbulence, Coherent Structures, Dynamical Systems and Symmetry, Cambridge University Press, Cambridge, 1996.
[10] J.M. Hyman, B. Nicolaenko, S. Zaleski, Order and complexity in the Kuramoto–Sivashinski model of weakly turbulent interfaces, Physica D 23 (1986) 265–292.
[11] C. Kane, J.E. Marsden, M. Ortiz, M. West, Variational integrators and the Newmark algorithm for conservative and dissipative mechanical systems, Int. J. Numer. Methods Eng., in press.
[12] M. Kirby, D. Armbruster, Reconstructing phase space from PDE simulations, Z. Angew. Math. Phys. 43 (1992) 999–1022.
[13] S. Lall, P. Krysl, J.E. Marsden, Structure-preserving model reduction of mechanical systems, preprint.
[14] J.E. Marsden, Lectures on Mechanics, London Mathematical Society Lecture Note Series 174, Cambridge University Press, Cambridge, 1992.
[15] J.E. Marsden, R. Montgomery, T.S. Ratiu, Reduction, symmetry, and phases in mechanics, Mem. Am. Math. Soc. 436 (1990) 1–110.
[16] J.E. Marsden, J. Ostrowski, Symmetries in motion: geometric foundations of motion control, Nonlinear Sci. Today, 1998 (http://link.springer-ny.com).
[17] J.E. Marsden, T.S. Ratiu, Introduction to mechanics and symmetry, Texts in Applied Mathematics, 2nd Edition, Vol. 17, Springer, Berlin, 1994.
[18] J.E. Marsden, T. Ratiu, J. Scheurle, Reduction theory and the Lagrange–Routh equations, J. Math. Phys., in press.
[19] H. Park, L. Sirovich, Turbulent thermal convection in a finite domain. II. Numerical results, Phys. Fluids A 2 (9) (1990) 1659–1668.
[20] L. Sirovich, Turbulence and the dynamics of coherent structures. I-III, Quart. Appl. Math. XLV (3) (1987) 561–590.
[21] L. Sirovich, H. Park, Turbulent thermal convection in a finite domain. I. Theory, Phys. Fluids A 2 (9) (1990) 1649–1658.
[22] L. Sirovich, J.D. Rodriguez, Coherent structures and chaos: a model problem, Phys. Lett. A 120 (3) (1987) 211–214.
[23] N. Smaoui, D. Armbruster, Symmetry and the Karhunen–Loève analysis, SIAM J. Sci. Comput. 18 (5) (1997) 1526–1532.