Exam #2, March 11, 2015
Math 527, University of New Hampshire

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Section: all

READ ALL INSTRUCTIONS CAREFULLY

1. Write your name and section number **on each page.**
2. Correct and legible name and section are worth 2 pts of each problem.
3. Show your work and put a box or circle around your answers.
4. Partial credit will be given only if your work is written clearly and in equations.

Problem 1: JUDICIOUS GUESSING. (40 pts) Find the general solution of the differential equation using the method of judicious guessing (also known as undetermined coefficients). Differentiation is with respect to x.

$$y'' - 6y' + 9y = x + xe^{3x}$$

homog prob $\Rightarrow$ $\lambda^2 - 6\lambda + 9 = 0$ repeated roots $\lambda = 3$

homog soln $y_1 = e^{3x}$ $y_2 = x e^{3x}$

break RHS into two parts

for $y'' - 6y' + 9y = x$ guess $y_{p1} = Ax + B$, so

$y_{p1}' = A$

$y_{p1}'' = 0$

so $-6A + 9(Ax+B) = x$

x terms $\Rightarrow$ $A = 1/9$, 1 terms $\Rightarrow$ $-6\frac{1}{9} + \cancel{x} + 9B = \cancel{x}$

$9B = \frac{2}{3}$, $B = \frac{2}{27}$

$y_{p1} = \frac{1}{9}x + \frac{2}{27}$

for $y'' - 6y' + 9y = xe^{3x}$ would normally guess $y_{p2} = (Ax + B)e^{3x}$

but both terms in that are part of homog. soln, so multiply by x^2.

and guess

$$y_{p2} = (Ax^3 + Bx^2)e^{3x}$$

$$y_{p2}' = (3Ax^2 + 2Bx)e^{3x} + (3Ax^3 + 3Bx^2)e^{3x}$$

$$= \left[3Ax^3 + 3(A+B)x^2 + 2Bx\right]e^{3x}$$

$$y_{p2}'' = \left[9Ax^2 + 6(A+B)x + 2B\right]e^{3x}$$

$$+ \left[9Ax^3 + 9(A+B)x^2 + 6Bx\right]e^{3x}$$

$$y_{p2}'' = \left[9Ax^3 + (18A+9B)x^2 + (6A+12B)x + 2B\right]e^{3x}$$

plugging those into $y'' - 6y' + 9y = xe^{3x}$ and dividing by e^{3x} gives

$$\cancel{9Ax^3} + (\cancel{18A} + \cancel{9B})x^2 + (6A + \cancel{12B})x + 2B - \cancel{18Ax^3} - 18(\cancel{A} + \cancel{B})x^2 - \cancel{12Bx} + \cancel{9Ax^3} + \cancel{9Bx^2} = x$$

x terms $\Rightarrow$ $6A = 1$ $\quad A = \frac{1}{6}$

1 terms $\Rightarrow$ $B = 0$

so $y_{p2} = \frac{1}{6}xe^{3x}$

gen'l soln $\quad y(x) = c_1 y_1(x) + c_2 y_2(x) + y_{p1}(x) + y_{p2}(x)$

$$\boxed{y(x) = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{9}x + \frac{2}{27} + \frac{1}{6}xe^{3x}}$$

Note: this problem was harder than intended. We graded generously for y_{p2}.

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Problem 2: VARIATION OF PARAMETERS. (40 pts) Find the general solution of the differential equation using variation of parameters. Differentiation is with respect to x.

$$y'' - y' - 6y = 2e^{2x}$$

homog prob $\Rightarrow$ $\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0 \quad \lambda = 3, -2$

$y_1(x) = e^{3x}, \quad y_2(x) = e^{-2x}$

$y_1' = 3e^{3x} \quad y_2' = -2e^{-2x}$

$$W = \begin{vmatrix} e^{3x} & e^{-2y} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -2e^{x} - 3e^{x} = -5e^{x}$$

$$u_1' = \frac{1}{-5e^{x}} \begin{vmatrix} 0 & e^{-2x} \\ 2e^{2x} & -2e^{-2y} \end{vmatrix} = -\frac{1}{5} e^{-x}(-2) \qquad u_1(x) = -\frac{2}{5}\int e^{-x}\,dx = -\frac{2}{5}e^{-x}$$

$$u_2' = \frac{1}{-5e^{x}} \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & 2e^{2x} \end{vmatrix} = -\frac{2}{5} e^{4x} \qquad u_2(x) = \frac{-2}{5}\int e^{4x}\,dx = -\frac{1}{10}e^{4x}$$

gen'l soln

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$= c_1 e^{3x} + c_2 e^{-2x} - \frac{2}{5}e^{2x} - \frac{1}{10}e^{2x}$$

$$\boxed{y(x) = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{2}e^{2x}}$$

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Problem 3: SHORT ANSWERS. (20 pts)

(a) (5 pts) What property must the operator L have in order to be a *linear* operator?

$$L(\alpha f + \beta g) = \alpha L f + \beta L g$$

(where α, β are consts and f, g are functions)

(b) (5 pts) Give an example of a second-order linear differential operator.

$$L = 9\frac{d^2}{dx^2} + 2\frac{d}{dx} - 1$$

(or simplest, $L = \frac{d^2}{dx^2}$)

(c) (5 pts) Let L be a linear differential operator. Use your answer to (a) to show that if y_{p_1} is a particular solution to the equation $Ly = g_1$ and y_{p_2} is a particular solution to the equation $Ly = g_2$, then $y_p = y_{p_1} + y_{p_2}$ is a particular solution to the equation $Ly = g_1 + g_2$.

$$L y_p = L(y_{p_1} + y_{p_2})$$
$$= L y_{p_1} + L y_{p_2} \quad \text{by linearity}$$

$$L y_p = g_1 + g_2 \quad \blacksquare$$

(d) (5 pts) How can the property discussed in (c) help solve a nonhomogeneous linear differential equation? Answer in your own words, in a complete sentence.

The property (c) allows you to break up nonhomog. problems of the form $Ly = g_1 + g_2$ into separate problems $Ly = g_1$ and $Ly = g_2$ and solve them separately.