(10 pts) 9. $t\frac{d^2y}{dt^2} - (1+3t)\frac{dy}{dt} + 3y = 0$, using the ansatz $y(t) = e^{\lambda t}$

$\Rightarrow t\cdot\lambda^2 e^{\lambda t} - (1+3t)\lambda e^{\lambda t} + 3e^{\lambda t} = 0$, $[t\lambda^2 - (1+3t)\lambda + 3]e^{\lambda t} = 0$

so $t\lambda^2 - (1+3t)\lambda + 3 = 0$, $\lambda = \frac{(1+3t)\pm\sqrt{(1+3t)^2 - 4\cdot t\cdot 3}}{2\cdot t} = \frac{(1+3t)\pm(1-3t)}{2t} = 3$ or $\frac{1}{t}$

if $y = e^{3t}$, $\Rightarrow [9t - (1+3t)\cdot 3 + 3]e^{\lambda t} = 0$ ✓

when $y = e^{\frac{1}{t}\cdot t} = e^1 = e$ $\Rightarrow t\cdot 0 - (1+3t)\cdot 0 + 3e = 0$ X

so we already got one known solution $y_1 = e^{3t}$

Assume $y_2 = u(t)\cdot y_1$, transform the original Eqn into the standard form,

$\frac{d^2y}{dt^2} - (3+\frac{1}{t})y' + \frac{3}{t}y = 0$, $P(t) = -(3+\frac{1}{t})$

$u' = \frac{C_1 e^{-\int P dt}}{y_1^2} = \frac{C_1 e^{\int(3+\frac{1}{t})dt}}{e^{6t}} = C_1 e^{-6t}\cdot e^{3t+\ln|t|} = C_1 e^{-3t}|t|$

$u = C_1\int e^{-3t}|t|\,dt = C\int e^{-3t}t\,dt = -\frac{C}{3}\int t\,de^{-3t}$

$= -\frac{C}{3}[te^{-3t} - \int e^{-3t}dt] = -\frac{C}{3}[te^{-3t} + \frac{1}{3}e^{-3t}]$

$y_2 = u\cdot y_1 = -\frac{C}{3}[t + \frac{1}{3}]$, it satisfies the original ODE.

$\therefore y = C_1 e^{3t} + C_2(t + \frac{1}{3})$

(10 pts)

10. $t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$, using the ansatz $y(t) = t^\lambda$

$\Rightarrow t^2\cdot\lambda(\lambda-1)t^{\lambda-2} + \alpha\cdot t\cdot\lambda t^{\lambda-1} + \beta t^\lambda = 0$

$\Rightarrow [\lambda(\lambda-1) + \alpha\lambda + \beta]t^\lambda = 0$, so $\lambda^2 + (\alpha-1)\lambda + \beta = 0$ (+4 pts)

$\lambda = \frac{1-\alpha\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$ (assume $(\alpha-1)^2 - 4\beta > 0$)

$\lambda_1 = \frac{1-\alpha+\sqrt{(\alpha-1)^2-4\beta}}{2}$, $\lambda_2 = \frac{1-\alpha-\sqrt{(\alpha-1)^2-4\beta}}{2}$ (+4 pts)

Then $y(t) = C_1 t^{\lambda_1} + C_2 t^{\lambda_2}$ (+2 pts)