6. $y'' + 2y' + 5y = 0$; $y(0) = 0$, $y'(0) = 2$ (10 pts)

Find the roots of the auxiliary equation $\lambda^2 + 2\lambda + 5 = 0$

$(\lambda + 1)^2 + 4 = 0 \Rightarrow \lambda = -1 \pm 2i$ $\therefore$ $y = e^{-x} \cdot (C_1 \cos 2x + C_2 \sin 2x)$

$y(0) = C_1 = 0$ Then $y = e^{-x} \cdot C_2 \sin 2x$, $y' = -C_2 e^{-x} \sin 2x + 2C_2 e^{-x} \cos 2x$

$y'(0) = 2C_2 = 2 \Rightarrow C_2 = 1$, $y = e^{-x} \sin 2x$

(10 pts)

7. Proof. (1) $y'' + w^2 y = 0$, we get the auxiliary Eqn. $\lambda^2 + w^2 = 0$

$\therefore \lambda = \pm wi$, $y = C_1 e^{iwt} + C_2 e^{-iwt}$

To show e^{iwt} and e^{-iwt} are linearly independent,

if and only if $C_1 = C_2 = 0$, we have $C_1 e^{iwt} + C_2 e^{-iwt} = 0$

Assume $C_1 \neq 0$, $\Rightarrow e^{iwt} = -\frac{C_2}{C_1} e^{-iwt}$, By Euler's formula,

$\cos wt + i \sin wt = -\frac{C_2}{C_1} [\cos wt - i \sin wt]$

which means $\frac{-C_2}{C_1} = 1$ and $\frac{C_2}{C_1} = 1$ by compare the coefficients of the real parts and the imaginary parts.

That's impossible. $\Leftrightarrow C_1 = C_2 = 0$, we have $C_1 e^{iwt} + C_2 e^{-iwt} = 0$

(2) $y = C_1 e^{iwt} + C_2 e^{-iwt}$

$= C_1 (\cos wt + i \sin wt) + C_2 (\cos wt - i \sin wt)$

$= (C_1 + C_2) \cos wt + (C_1 i - C_2 i) \sin wt$

$= \tilde{C}_1 \cos wt + \tilde{C}_2 \sin wt$, where $\tilde{C}_1 = C_1 + C_2$, $\tilde{C}_2 = C_1 i - C_2 i$

(10 pts)

8. Proof. $(\cos x + i \sin x)^n = (e^{ix})^n = e^{inx} = \cos nx + i \sin nx$

$n = 2$, $(\cos x + i \sin x)^2 = \cos^2 x - \sin^2 x + i\, 2 \sin x \cos x = \cos 2x + i \sin 2x$

so $\cos 2x = \cos^2 x - \sin^2 x$, $\sin 2x = 2 \sin x \cos 2x$