Diff Eqs HW2 Solutions.

1. $2x\sin y + y^3e^x + (x^2\cos y + 3y^2e^x)\frac{dy}{dx} = 0$

multiply by dx first:

$$\underbrace{(2x\sin y + y^3e^x)}_{A}dx + \underbrace{(x^2\cos y + 3y^2e^x)}_{B}dy = 0$$

See if it's exact:

$$\frac{\partial A}{\partial y} = 2x\cos y + 3y^2e^x = \frac{\partial B}{\partial x} = 2x\cos y + 3y^2e^x$$

therefore the eqn is exact.
the solution (A family of solutions) has the form $F(x,y) = C$, (C a constant).

need to find $F(x,y)$.

first write: $\frac{\partial F}{\partial x} = 2x\sin y + y^3e^x$

think of y as a constant and integrate:

$$F = \int 2x\sin y + y^3e^x\,dx = \frac{2x^2}{2}\sin y + y^3e^x + g(y)$$

next find $g(y)$ using:

$$\frac{\partial F}{\partial y} = B$$

$$\Rightarrow x^2\cos y + 3y^2e^x + g'(y) = x^2\cos y + 3y^2e^x \Rightarrow$$

$$g'(y) = 0 \Rightarrow g(y) = c_1 \quad (c_1 \text{ const})$$

$$\Rightarrow F(x,y) = x^2\sin y + y^3e^x + c_1$$

$$\Rightarrow \text{solution:} \quad x^2\sin y + y^3e^x = C$$

2) $1 + (1+xy)e^{xy} + (1+x^2 e^{xy})\dfrac{dy}{dx} = 0$

multiply by dx:

$$\underbrace{[1+(1+xy)e^{xy}]}_{A}\,dx + \underbrace{(1+x^2e^{xy})}_{B}\,dy = 0$$

See if it's exact:

$$\frac{\partial A}{\partial y} = xe^{xy} + xe^{xy} + xy \cdot xe^{xy} = 2xe^{xy} + x^2ye^{xy}$$

$$\frac{\partial B}{\partial x} = 2xe^{xy} + ye^{xy} \cdot x^2$$

$\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x} \Rightarrow$ the eqn is exact.

next find the solution $F(x,y) = C$.

first write

$\dfrac{\partial F}{\partial y} = 1 + x^2e^{xy}$, think of x as a constant

and integrate w.r.t y:

$$F = \int 1 + x^2e^{xy}\,dy = y + x^2 \cdot \frac{1}{x}e^{xy} + g(x$$

next find $g(x)$ using

$$\frac{\partial F}{\partial x} = A \Rightarrow$$

$$e^{xy} + xye^{xy} + g'(x) = 1 + e^{xy} + xye^{xy}$$

$g'(x) = 0 \Rightarrow g(x) = c_1$ (c_1 a const)

$\Rightarrow F(x,y) = y + xe^{xy} + c_1$

$\Rightarrow$ solution: $y + xe^{xy} = C$

3) $y\sec^2 t + \sec t\tan t + (2y + \tan t)\frac{dy}{dt} = 0$

multiply by dt:

$$\underbrace{(y\sec^2 t + \sec t\tan t)}_{A}dt + \underbrace{(2y + \tan t)}_{B}dy = 0$$

test for exactness:

$\frac{\partial A}{\partial y} = \sec^2 t = \frac{\partial B}{\partial t} = \sec^2 t \Rightarrow$ the eqn is exact.

find a family of solutions of form $F(t,y) = C$

first write $\frac{\partial F}{\partial y} = 2y + \tan t$

think of t as a constant and integrate w.r.t y:

$$F = \int 2y + \tan t \; dy = \frac{2y^2}{2} + \tan t \cdot y + g(t)$$

$$= y^2 + y\tan t + g(t)$$

next find $g(t)$ using:

$\frac{\partial F}{\partial t} = A \Rightarrow$

$\cancel{y\sec^2 t} + g'(t) = \cancel{y\sec^2 t} + \sec t\tan t \Rightarrow$

$g'(t) = \sec t\tan t \Rightarrow$

$g(t) = \int \sec t\tan t\; dt = \int \frac{1}{\cos t}\cdot\frac{\sin t}{\cos t}\,dt = \int \frac{\sin t\, dt}{\cos^2 t}$

$= \int (\cos t)^{-2}\sin t\, dt$, $\begin{cases} \text{let } u = \cos t \\ du = -\sin t\, dt \end{cases} \Rightarrow$

$-\int u^{-2}\,du = -\frac{u^{-1}}{-1} = \frac{1}{u} = \frac{1}{\cos t} = \sec t$

so $F(t,y) = y^2 + y\tan t + \sec t + C_1$

and solution: $y^2 + y\tan t + \sec t = C$

4. $y - ye^x + (y - e^x)\dfrac{dy}{dx} = 0$

$$\underbrace{(y - ye^x)}_{A}dx + \underbrace{(y - e^x)}_{B}dy = 0$$

test for exactness:

$$\frac{\partial A}{\partial y} = 1 - e^x \neq \frac{\partial B}{\partial x} = -e^x \Rightarrow \text{NOT exact.}$$

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5. $x - y^3 + y^2\sin x - (3xy^2 + 2y\cos x)\dfrac{dy}{dx} = 0$

$$\underbrace{(x - y^3 + y^2\sin x)}_{A}dx \underbrace{- (3xy^2 + 2y\cos x)}_{B}dy = 0$$

$$\frac{\partial A}{\partial y} = -3y^2 + 2y\sin x = \frac{\partial B}{\partial x} = -3y^2 + 2y\sin x$$

$\Rightarrow$ the equation is exact.

find a family of solutions of the form $F(x,y) = C$.

write $\dfrac{\partial F}{\partial x} = x - y^3 + y^2\sin x$, think of y as a constant

and integrate w.r.t x:

$$F = \int x - y^3 + y^2\sin x\,dx = \frac{x^2}{2} - y^3x - y^2\cos x + g(y)$$

next find $g(y)$ using:

$$\frac{\partial F}{\partial y} = B \Rightarrow$$

$-3y^2x - 2y\cos x + g'(y) = -3xy^2 - 2y\cos x \Rightarrow$

$g'(y) = 0 \Rightarrow g(y) = c_1 \Rightarrow$

$F(x,y) = \frac{x^2}{2} - y^3x - y^2\cos x + c_1$

ant solution: $\frac{x^2}{2} - y^3x - y^2\cos x = C$

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6) $ty^3 + 3t^2y^2 \frac{dy}{dt} = 0 \qquad y(1) = 1$

$\underbrace{ty^3}_{A} dt + \underbrace{3t^2y^2}_{B} dy = 0$

test for exactness:

$\frac{\partial A}{\partial y} = 3y^2t \neq \frac{\partial B}{\partial t} = 6ty^2$ NOT exact

using a different method we get:

$\frac{dy}{dt} = \frac{-ty^3}{3t^2y^2} \Rightarrow \frac{dy}{dt} = \frac{-y}{3t} \rightarrow$ separable

$-\frac{1}{y}dy = \frac{dt}{3t} \Rightarrow$

$-\int \frac{1}{y}dy = \frac{1}{3}\int \frac{dt}{t} \Rightarrow$

$-\ln|y| = \frac{1}{3}\ln t + c_1$ or

$\ln|y| = -1/3 \ln t + c_2$ (where $c_2 = -c_1$)

$\Rightarrow e^{\ln|y|} = e^{-1/3\ln t + c_2} = e^{-1/3\ln t} \cdot e^{c_2}$

let $(e^{c_2} = c_3) \Rightarrow$

$|y| = c_3 t^{-1/3}$ or $y = \pm c_3 \frac{1}{\sqrt[3]{t}}$

let $\pm c_3 = \tilde{c} \Rightarrow y = \tilde{c}\frac{1}{\sqrt[3]{t}}$

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then apply the initial condition $y(1)=1$.

$y(1) = \tilde{c}\,\frac{1}{\sqrt{1}} = \tilde{c} = 1 \Rightarrow \boxed{\tilde{c}=1}$

$\Rightarrow y(t) = \frac{1}{\sqrt[3]{t}}$

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7) $2x\cos y + 3x^2y + (x^3 - x^2\sin y - y)\frac{dy}{dx} = 0$

$\underbrace{(2x\cos y + 3x^2y)}_{A}\,dx + \underbrace{(x^3 - x^2\sin y - y)}_{B}\,dy = 0 \Rightarrow$

test for exactness:

$\frac{\partial A}{\partial y} = -2x\sin y + 3x^2 = \frac{\partial B}{\partial x} = 3x^2 - 2x\sin y \Rightarrow$

the equation is exact.

find a family of solutions of the form $F(x,y) = C$.

write $\frac{\partial F}{\partial x} = 2x\cos y + 3x^2y \Rightarrow$

think of y as a constant and integrate w.r.t x.

$f = \int 2x\cos y + 3x^2y\,dx = \frac{2x^2}{2}\cos y + \frac{3x^3}{3}y + g(y)$

$= x^2\cos y + x^3y + g(y)$

next find $g(y)$ using

$\frac{\partial f}{\partial y} = B \Rightarrow$

$$-\cancel{x^2\sin y} + \cancel{x^3} + g'(y) = \cancel{x^3} - \cancel{x^2\sin y} - y \Rightarrow$$

$$g'(y) = -y \Rightarrow g(y) = -y^2/2 + c_1 \Rightarrow$$

$$F(x,y) = x^2\cos y + x^3 y - y^2/2 + c_1$$

$\Rightarrow$ Solution: $$x^2\cos y + x^3 y - y^2/2 = C$$