Homework #1 **IAM 961, UNH fall 2015**
Due Monday, Sept 21th in lecture

1. Prove that any linear map $\mathcal{L} : \mathbb{C}^n \to \mathbb{C}^m$ can written as an $m \times n$ matrix. You can assume the existence of an orthogonal basis for both spaces.

2. Prove that $\|AB\|_p \leq \|A\|_p \|B\|_p$.

3. If u and v are m-vectors the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the indentity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is null(A)? (Trefethen exercise 2.6).

4. Let $A \in \mathbb{C}^{m\times m}$ be Hermitian. An eigenvector of A is a nonzero vector x such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue. Prove (a) that all the eigenvalues of A are real and (b) that if x and y are eigenvectors with distinct eigenvalues, then they are orthogonal. (Trefethen exercise 2.3).

5. Use the SVD to prove that any matrix in $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is dense subset of $\mathbb{C}^{m\times n}$. Use the 2-norm in your proof. (Trefethen exercise 5.2.)

6. Visit `www.julialang.org`, and download and install Julia on a computer. You can download a precompiled executable, or you can download the source code (using the "git" source-code control system) and compile it. Read enough documentation that you can do the following

   1. construct a random $m \times m$ matrix A and a random m-dim vector x for $m = 4$
   2. compute $b = Ax$
   3. solve $A\hat{x} = b$ numerically for $\hat{x}$, using the backslash operator \
   4. compute the relative error $\|x - \hat{x}\|/\|x\|$ where $\|\cdot\|$ is the 2-norm

Print out and turn in your results for these steps. You'll surely make errors while figuring this out. I don't need to see those errors, so clean up your results before printing. Repeat a few times for higher-dimensional problems, say $m = O(10)$ and $m = O(100)$.

These problems are intended to develop or exercise your understanding of basis sets, matrix norms, and singular versus nonsingular matrices. Problems 1, 2, and 4 are representative of the kind of analysis we'll need for the development of our main numerical linear algebra algorithms. Note that a couple of these are proofs are outlined verbally in the text. You can look at these proofs and follow the general strategy. But see if you can improve on the presentation.