Homework #1 **IAM 961, UNH fall 2014**
Due Monday, Sept 22th in lecture

1. Prove that any linear map $\mathcal{L} : \mathbb{C}^n \to \mathbb{C}^m$ can written as an $m \times n$ matrix. Strategy: Assume the existence of the canonical basis sets $\{e_1, e_2, \ldots\}$ where $e_1 = [1\ 0\ 0 \ldots 0]^*$, etc. for both $\mathbb{C}^m$ and $\mathbb{C}^n$. Then derive a matrix L that maps the canonical-basis coordinates of an arbitrary vector $x \in \mathbb{C}^n$ to the canonical-basis coordinates of $y = \mathcal{L}x$, making use of the fact that $\mathcal{L}$ acts linearly on the expansion of x in the canonical basis.

2. Prove that $\|AB\|_p \leq \|A\|_p \|B\|_p$.

3. Prove that $\|A\|_1 = \max_{1 \leq j \leq n} \|a_j\|_1$ where $\{a_j\}$ are the columns of the $A \in \mathbb{C}^{m \times n}$.

4. If u and v are m-vectors the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the indentity.* Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is null(A)? (Trefethen exercise 2.6).

5. Let $A \in \mathbb{C}^{m \times m}$ be Hermitian. An eigenvector of A is a nonzero vector x such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue. Prove (a) that all the eigenvalues of A are real and (b) that if x and y are eigenvectors with distinct eigenvalues, then they are orthogonal. (Trefethen exercise 2.3).

6. Use the SVD to prove that any matrix in $\mathbb{C}^{m \times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is dense subset of $\mathbb{C}^{m \times n}$. Use the 2-norm in your proof. (Trefethen exercise 5.2. This one sounds fancy but is very simple.)

These problems are intended to develop or exercise your understanding of basis sets, matrix norms, and singular versus nonsingular matrices. Problems 1, 2, and 4 are representative of the kind of analysis we'll need for the development of our main numerical linear algebra algorithms.

Note that a couple of these are proofs are outlined verbally in the text. You can look at these proofs and follow the general strategy. But see if you can improve on the presentation.