# COHERENT STRUCTURES IN TURBULENCE: A DYNAMICAL-SYSTEMS PERSPECTIVE

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*Summary* Recent theoretical and numerical work has revealed a large number of 3D, fully nonlinear, unstable solutions of the Navier-Stokes equations which capture essential features of classically observed coherent structures in unsteady and turbulent flows. These solutions take the form of equilibria (steady states), traveling waves, and periodic orbits; they have been discovered in plane Couette, pipe, channel, and isotropic flows, among others [1, 2, 3, 4, 5, 6]. The unstable solutions lie within the invariant measure of turbulent flow, thus specifying where turbulence lives within its infinite-dimensional state space. On the other hand, many solutions have remarkably low-dimensional unstable manifolds, so that the dynamics of turbulence within the invariant set can be understood as a series of transitions between solutions, along a low-dimensional network of connections formed by the intersections of their stable and unstable manifolds. As such, these unstable solutions provide a long-hoped-for bridge between turbulence and dynamical systems theory and a practical strategy for determining and exploiting the inherent low dimensionality of unsteady and moderately turbulent dynamics.

## INTRODUCTION

In moderate-Reynolds plane Couette flow, the dominant large-scale coherent structures are pairs of counter-rotating vortices that convect fluid from the moving walls and create alternating high and low-speed streaks at the midplane. This fundamental roll-streak structure is evident in the Nagata-Busse-Clever-Waleffe equilibrium solution of plane Couette flow [1, 3].

## RESULTS

### State-Space Portraits: Unstable Solutions Turbulent Plane Couette Flow

We have computed a large number of additional equilibrium and traveling wave solutions of plane Couette flow which exhibit roll-streak structures with a variety of symmetries and scales [7]. As these solutions lie within the invariant measure of the turbulent flow and replicate its important structures, they form convenient basis sets for producing low-dimensional state space portraits of turbulent dynamics.

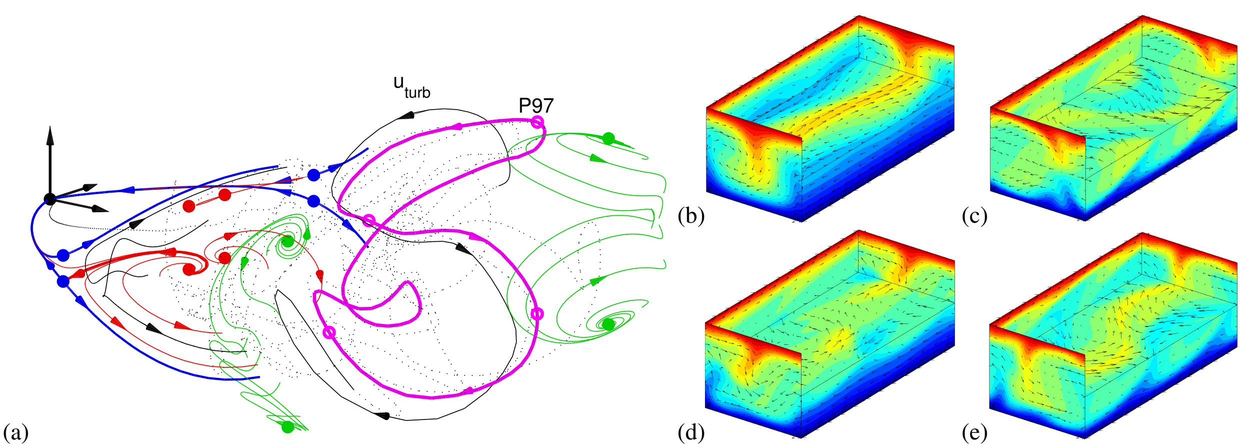


**Figure 1. (a) A state-space portrait of turbulent plane Couette flow.** Symmetry-related equilibria (solid dots) and their unstable manifolds (thin solid lines), a heteroclinic connection between two equilibria (thick red line), and a turbulent trajectory (solid black lines, $\mathbf{u}_{turb}$) shadowing a periodic orbit (thick magenta loop, P97). The black dot at the origin is the stable laminar equilibrium. **(b-e) Velocity fields along an unstable periodic orbit** at intervals $\Delta t = 25$ marked by open magenta dots in (a), starting at the point labeled P97.

Figure 1 shows such a state space portrait for plane Couette flow at $Re = 400$ in a small periodic box. The state-space coordinates are given by the $L^2$ inner product of the time-evolving velocity field $\mathbf{u}(t)$ against a 3d basis set formed by orthogonalizing four symmetry-related equilibrium solutions of the flow (blue dots). Solid dots represent equilibrium solutions of the flow, and solid lines indicate trajectories in the unstable manifolds of the equilibria, computed with fully-resolved direct numerical simulations.

The state-space portrait reveals how the *turbulent flow is organized by the equilibrium and periodic-orbit solutions and*

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*their low-dimensional unstable manifolds.* Turbulent trajectories (dotted lines) do not wander randomly, but instead inhabit the same region of phase space as the unstable solutions. Segments of the turbulent trajectory (solid black lines) clearly shadow the low-dimensional unstable manifolds of the flow's equilibrium solutions, and at these times, the turbulent velocity field closely resembles the nearby equilibrium solution. Turbulent trajectories can also be observed to shadow periodic orbits. For example, the portion of the turbulent trajectory labeled $\mathbf{u}_{turb}$ follows closely the periodic orbit labeled P97. During this time, the turbulent velocity field is virtually indistinguishable from the velocity field of the periodic orbit. Eventually, the turbulent trajectory moves away from the orbit, but not in a random way; instead, the turbulent trajectory moves away along an unstable manifold of the orbit to the neighborhood of another solution. This process repeats itself as the turbulent flow continues to evolve in time. Thus the picture illustrates two key concepts: (1) *Coherent structures in turbulence are the physical images of close passes to unstable solutions of the flow,* and (2) *turbulent dynamics can be understood as a series of transitions between nearby unstable invariant states*.

### Approximating Turbulent Attractors with Periodic Orbits

Further, since the periodic orbits of a chaotic dynamical system are dense in the system's invariant measure, the periodic orbits of a turbulent flow provide a way to approximate its attracting set. Figure 2 compares the state-space visualizations of a set of some forty unstable periodic orbits of plane Couette flow (in a certain invariant symmetric subspace) and the flow's natural measure, represented by a typical trajectory in the same invariant subspace [8]. It is evident that the turbulent flow explores the region of state-space covered by the set of periodic orbits. The correspondence between the set of orbits and the system's natural measure is made precise by Periodic Orbit Theory, which relates statistical averages over invariant measures of chaotic systems to weighted sums over the infinite set of its unstable periodic orbits.

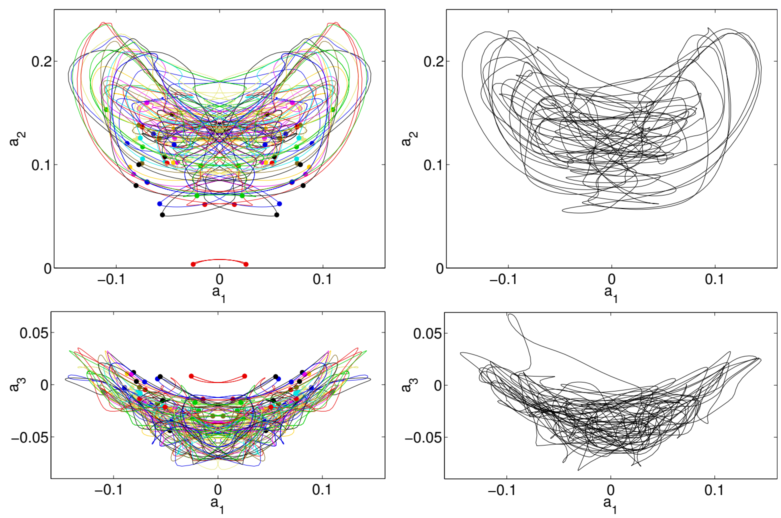


**Figure 2.** State-space visualization of (left) a set of periodic orbits of turbulent plane Couette flow compared to (right) the natural measure as explored by a generic turbulent trajectory. The projection is onto the first three principal components of the dynmically dominant periodic orbit and its symmetric counterpart.

## CONCLUSIONS

We have computed a large number of equilibrium, traveling wave, and periodic orbit solutions of plane Couette flow and used these solutions to construct state-space portraits of the turbulent flow's dynamics. The state-space portraits reveal a rich set of dynamical phenomena in the turbulent flow, including heteroclinic connections between equilibrium solutions [9], and turbulent trajectories shadowing periodic orbits and the unstable manifolds of equilibria. Coherent structures in the turbulent flow are revealed as close passes to weakly unstable solutions. Further, we present numerical evidence that the turbulent flow explores the same region of state space inhabited by a large set of its periodic orbits.

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