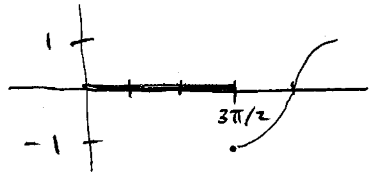
MATH 527
Spring 2015
HW #9 SOLNS

1) $f(t) = \begin{cases} 0, & 0 \le t < \frac{3\pi}{2} \\ \sin t, & \frac{3\pi}{2} \le t \end{cases}$



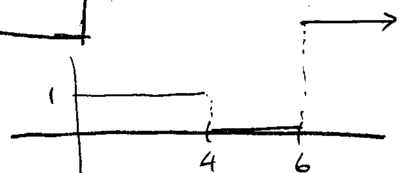
$$\boxed{f(t) = \sin(t)\,\mathcal{U}\left(t - \tfrac{3\pi}{2}\right)}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\sin(t)\,\mathcal{U}\left(t-\tfrac{3\pi}{2}\right)\right\} = e^{-\frac{3\pi}{2}s}\,\mathcal{L}\left\{\sin(t)\big|_{t \to t+\frac{3\pi}{2}}\right\}$$

$$= e^{-\frac{3\pi}{2}s}\,\mathcal{L}\left\{\sin\left(t+\tfrac{3\pi}{2}\right)\right\} = e^{-\frac{3\pi}{2}s}\,\mathcal{L}\{-\cos(t)\}$$

$$\boxed{\therefore\ \mathcal{L}\{f(t)\} = e^{-\frac{3\pi}{2}s}\left(\frac{-s}{s^2+1}\right)}$$

2). $f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & 4 \le t < 6 \\ 3, & 6 \le t \end{cases}$



$$\boxed{f(t) = 1 - \mathcal{U}(t-4) + 3\,\mathcal{U}(t-6)}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{\mathcal{U}(t-4)\cdot 1\} + 3\,\mathcal{L}\{\mathcal{U}(t-6)\cdot 1\}$$

$$= \frac{1}{s} - e^{-4s}\,\mathcal{L}\{1|_{t\to t+4}\} + 3\,\mathcal{L}\{1|_{t \to t+6}\}\,e^{-6s}$$

$$= \frac{1}{s} - e^{-4s}\,\mathcal{L}\{1\} + 3e^{-6s}\,\mathcal{L}\{1\}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s} - e^{-4s}\left(\frac{1}{s}\right) + 3e^{-6s}\left(\frac{1}{s}\right)}$$

3) $y'+y=f(t),\ y(0)=0 \quad f(t)=\begin{cases}1 & 0\le t<3\\ -1 & 3\le t\end{cases}$

$$y'+y=1-2\mathcal{U}(t-3)$$

$$sY(s)+0+Y(s)=\mathcal{L}\{1\}-2\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$(s+1)Y(s)=\frac{1}{s}-2e^{-3s}\mathcal{L}\{1\}$$

$$Y(s)=\left(\frac{1}{s}-\frac{2e^{-3s}}{s}\right)\frac{1}{s+1}=\frac{1}{s(s+1)}-\frac{2e^{-3s}}{s(s+1)}$$

$$\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}-2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)}\right\}$$

$$1=As+B(s+1)$$

$$s=0 \rightarrow 1=B$$

$$s=-1 \rightarrow -1=A$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{-1}{s+1}+\frac{1}{s}\right\}-2\mathcal{U}(t-3)\left[\mathcal{L}^{-1}\left\{\frac{-1}{s+1}+\frac{1}{s}\right\}\right]_{t\to t-3}$$

$$\boxed{y(t)=-e^{-t}+1+2\mathcal{U}(t-3)\left(-e^{-t+3}+1\right)}$$

4) $y''+4y=\sin(t)\mathcal{U}(t-2\pi) \quad y(0)=1,\ y'(0)=0$

$$s^2Y(s)-s(1)+0+4Y(s)=\mathcal{L}\{\sin(t)\mathcal{U}(t-2\pi)\}$$

$$(s^2+4)Y(s)-s=e^{-2\pi s}\mathcal{L}\{\sin(t)|_{t\to t+2\pi}\}$$

$$\text{"}\quad=e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$$

$$\text{"}\quad=e^{-2\pi s}\mathcal{L}\{\sin(t)\}$$

$$\text{"}\quad=e^{-2\pi s}\left(\frac{1}{s^2+1}\right)$$

$$Y(s)=\left(\frac{e^{-2\pi s}}{s^2+1}+s\right)\left(\frac{1}{s^2+4}\right)=\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}+\frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right\}+\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$=\mathcal{U}(t-2\pi)\left[\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}\right]_{t\to t-2\pi}+\cos(2t)$$

$$\frac{1}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

$$1=(As+B)(s^2+4)+(Cs+D)(s^2+1)$$

$$1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D) = 1 + 0s + 0s^2 + 0s^3$$

$$A+C=0 \rightarrow A=-C$$
$$B+D=0 \rightarrow B=-D$$
$$4A+C=0 \longrightarrow 4(-C)+C=0 \rightarrow C=0 \rightarrow A=0$$
$$4B+D=1 \longrightarrow 4(-D)+D=1 \rightarrow -3D=1 \rightarrow D=-\tfrac{1}{3}$$
$$B=\tfrac{1}{3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{U}(t-2\pi)\left[\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}\right\}\right]_{t\to t-2\pi} + \cos(2t)$$

$$y(t) = \mathcal{U}(t-2\pi)\left[\frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)\right]_{t\to t-2\pi} + \cos(2t)$$

$$y(t) = \mathcal{U}(t-2\pi)\left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\underbrace{\sin(2t-2\pi)}_{\sin(2(t-\pi))}\right] + \cos(2t)$$

$$\boxed{y(t) = \mathcal{U}(t-2\pi)\left[\frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)\right] + \cos(2t)}$$

5) $y'' - 5y' + 6y = \mathcal{U}(t-1)$, $y(0)=0$, $y'(0)=1$

$$s^2Y(s) - s(0) - 1 - 5(sY(s) - 0) + 6Y(s) = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$(s^2-5s+6)Y(s) - 1 = e^{-s}\mathcal{L}\{1|_{t\to t+1}\}$$

$$Y(s) = \left(\frac{e^{-s}}{s}+1\right)\frac{1}{(s^2-5s+6)} = \frac{e^{-s}}{s(s-3)(s-2)} + \frac{1}{(s-3)(s-2)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s-3)(s-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s-3)(s-2)}\right\}$$

$$\frac{1}{s(s-3)(s-2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-2}$$

$$1 = A(s-3)(s-2) + B(s)(s-2) + C(s)(s-3)$$

$$s=3 \rightarrow 1 = B(3)(1) \rightarrow \tfrac{1}{3} = B$$

$$s=2 \rightarrow 1 = C(2)(-1) \quad -\tfrac{1}{2} = C$$

$$s=0 \rightarrow 1 = A(-3)(-2) \quad \tfrac{1}{6} = A$$

$$\frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s-3)$$

$$s=2 \rightarrow 1 = B(-1) \rightarrow B=-1$$

$$s=3 \rightarrow 1 = A \rightarrow A=1$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{6s} + \frac{1}{3}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-2}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-3} - \frac{1}{s-2}\right\}$$

$$y(t) = \mathcal{U}(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{6s} + \frac{1}{3}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-2}\right\}\right]_{t \to t-1} + \quad '' \quad ''$$

$$y(t) = \mathcal{U}(t-1)\left[\frac{1}{6}(1) + \frac{1}{3}(e^{3t}) - \frac{1}{2}(e^{2t})\right]_{t \to t-1} + \left(e^{3t} - e^{2t}\right)$$

$$\boxed{y(t) = \mathcal{U}(t-1)\left(\frac{1}{6} + \frac{1}{3}e^{3(t-1)} - \frac{1}{2}e^{2(t-1)}\right) + e^{3t} - e^{2t}}$$

6) $y'' + 4y' + 5y = \delta(t-2\pi), \quad y(0) = y'(0) = 0$

$$s^2Y(s) - s(0) - (0) + 4(sY(s) - 0) + 5Y(s) = \mathcal{L}\{\delta(t-2\pi)\}$$

$$(s^2 + 4s + 5)Y(s) = e^{-2\pi s}$$

$$Y(s) = \frac{e^{-2\pi s}}{s^2 + 4s + 5} = \frac{e^{-2\pi s}}{(s+2)^2 + 1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s+2)^2 + 1}\right\} = \mathcal{U}(t-2\pi)\left[\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}\right]_{t \to t-2\pi}$$

$$y(t) = \mathcal{U}(t-2\pi)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s \to s+2}\right\}\right]_{t \to t-2\pi}$$

$$y(t) = \mathcal{U}(t-2\pi)\left[e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right]_{t \to t-2\pi}$$

$$y(t) = \mathcal{U}(t-2\pi)\left[e^{-2t}\sin(t)\right]_{t \to t-2\pi}$$

$$y(t) = \mathcal{U}(t-2\pi)\, e^{-2(t-2\pi)}\sin(t-2\pi)$$

$$\boxed{y(t) = \mathcal{U}(t-2\pi)\, e^{-2(t-2\pi)}\sin(t)}$$