Solutions of HW#8

1. Proof. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$\frac{d}{dx}\cos x = 0 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = -\sin x$$

2. Solutions (1) $y'' + k^2 y = 0$, ansatz $y = e^{\lambda x}$, $\lambda^2 e^{\lambda x} + k^2 e^{\lambda x} = 0$

$\therefore \lambda^2 + k^2 = 0$, $\lambda = \pm k i$, $y = C_1 \cos kx + C_2 \sin kx$

(2) Assume $y = \sum_{n=0}^{\infty} C_n x^n$, then $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

plug in the ODE, $\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + k^2 \sum_{n=0}^{\infty} C_n x^n = 0$

$\Rightarrow 2C_2 + \sum_{n=3}^{\infty} n(n-1) C_n x^{n-2} + k^2 \sum_{n=1}^{\infty} C_n x^n + k^2 C_0 = 0$.

shift $n=3 \to m=1$, $\therefore n-m=2$, $n=m+2$

Then $2C_2 + \sum_{m=1}^{\infty} (m+2)(m+1) C_{m+2} x^m + k^2 \sum_{m=1}^{\infty} C_m x^m + k^2 C_0 = 0$

$$\sum_{m=1}^{\infty} \left[(m+2)(m+1) C_{m+2} + k^2 C_m\right] x^m + 2C_2 + k^2 C_0 = 0$$

$$\begin{cases} 2C_2 + k^2 C_0 = 0 \\ (m+2)(m+1) C_{m+2} + k^2 C_m = 0, \quad m = 1, 2, \cdots \end{cases}$$

$\Rightarrow C_2 = -\frac{k^2}{2} C_0$, $m=1$, $6C_3 + k^2 C_1 = 0$, $C_3 = -\frac{k^2}{6} C_1$

$m=2$, $4 \cdot 3 \cdot C_4 + k^2 C_2 = 0 \Rightarrow C_4 = -\frac{k^2}{4 \cdot 3} C_2 = \frac{k^4}{4!} C_0$

$m=3$, $5 \cdot 4 \cdot C_5 + k^2 C_3 = 0 \Rightarrow C_5 = -\frac{k^2}{5 \cdot 4} C_3 = \frac{k^4}{6 \cdot 5 \cdot 4} C_1$

$m=4$, $6 \cdot 5 \cdot C_6 + k^2 C_4 = 0 \Rightarrow C_6 = -\frac{k^2}{6 \cdot 5} C_4 = -\frac{k^6}{6!} C_0$

$\vdots$

$$y = C_0 x^0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \cdots$$
$$= C_0 + C_1 x + \left(-\frac{k^2}{2} C_0\right) x^2 - \frac{k^2}{6} C_1 x^3 + \frac{k^4}{4!} C_0 x^4 + \frac{k^4}{6 \cdot 5 \cdot 4} C_1 x^5 - \frac{k^6}{6!} C_0 x^6 + \cdots$$
$$= C_0\left[1 - \frac{k^2}{2!} x^2 + \frac{k^4}{4!} x^4 - \frac{k^6}{6!} x^6 + \cdots\right] + C_1\left[x - \frac{k^2}{3!} x^3 + \frac{k^4}{5!} x^5 + \cdots\right]$$
$$= C_0 \cos kx + \frac{C_1}{k}\left[kx - \frac{k^3}{3!} x^3 + \frac{k^5}{5!} x^5 + \cdots\right], \quad \text{set } \frac{C_1}{k} = \tilde{C}_1$$
$$= C_0 \cos kx + \tilde{C}_1 \sin kx$$

3. Solution. $(x^2+1) y'' - 6y = 0$, assume $y = \sum_{n=0}^{\infty} C_n x^n$, $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

plug in, $(x^2+1) \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 6 \sum_{n=0}^{\infty} C_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) C_n x^n + \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 6 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\left[\sum_{n=2}^{\infty} n(n-1)C_n x^n + 1\cdot(1-1)C_1x^1 - 1\cdot(1-1)C_1x^1\right] + \left[\sum_{n=3}^{\infty} n(n-1)C_n x^{n-2} + 2C_2\right]$$
$$-6\left[\sum_{n=1}^{\infty} C_n x^n + C_0\right] = 0$$
$$\Rightarrow \sum_{n=1}^{\infty} n(n-1)C_n x^n + \sum_{n=3}^{\infty} n(n-1)C_n x^{n-2} + 2C_2 - 6\sum_{n=1}^{\infty} C_n x^n - 6C_0 = 0$$

shift $n=3 \rightarrow k=1$, $\therefore n-k=2$, $n=k+2$ (the other just switch $n \rightarrow k$)

$$\Rightarrow \sum_{k=1}^{\infty} k(k-1)C_k x^k + \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k - 6\sum_{k=1}^{\infty} C_k x^k + 2C_2 - 6C_0 = 0$$
$$\sum_{k=1}^{\infty}\left[k(k-1)C_k + (k+2)(k+1)C_{k+2} - 6C_k\right]x^k + 2C_2 - 6C_0 = 0$$

$$\therefore \begin{cases} 2C_2 - 6C_0 = 0 \\ k(k-1)C_k + (k+2)(k+1)C_{k+2} - 6C_k = 0, \quad k=1,2,3,\cdots \end{cases}$$

$\Rightarrow C_2 = 3C_0$, $k=1$, $3\cdot2\cdot C_3 - 6C_1 = 0$, $\therefore C_3 = C_1$

$k=2$, $2C_2 + 4\cdot3C_4 - 6C_2 = 0$, $C_4 = \frac{1}{3}C_2 = C_0$

$k=3$, $3\cdot2C_3 + 5\cdot4\cdot C_5 - 6C_3 = 0$, $C_5 = 0$

$k=4$, $4\cdot3\cdot C_4 + 6\cdot5\cdot C_6 - 6C_4 = 0$, $C_6 = -\frac{1}{5}C_4 = -\frac{1}{5}C_0$

$k=5$, $C_7 = -\frac{1}{3}C_5 = 0$

$k=6$, $C_8 = -\frac{3}{7}C_6 = \frac{3}{35}C_0$

$$y = C_0x^0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7 + C_8x^8 + \cdots$$
$$= C_0 + C_1x + 3C_0x^2 + C_1x^3 + C_0x^4 + 0\cdot x^5 - \frac{1}{5}C_0x^6 + 0 + \frac{3}{35}C_0x^8 + \cdots$$
$$= C_0\left[1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{35}x^8 + \cdots\right] + C_1\left[x + x^3 + 0 + 0 + \cdots\right]$$

4. Solution. $y'' - (x+1)y' - y = 0$, assume $y = \sum_{n=0}^{\infty} C_n x^n$, $y' = \sum_{n=1}^{\infty} nC_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - (x+1)\sum_{n=1}^{\infty} nC_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$$
$$\sum_{n=3}^{\infty} n(n-1)C_n x^{n-2} + 2C_2 - \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=1}^{\infty} nC_n x^{n-1} - \sum_{n=1}^{\infty} C_n x^n - C_1 = 0$$

shift $n=3 \rightarrow k=1$, $n-k=2$, $n=k+2$

$$\Rightarrow \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k - \sum_{k=1}^{\infty} kC_k x^k - \sum_{n=2}^{\infty} nC_n x^{n-1} - C_0 - \sum_{n=1}^{\infty} C_n x^n - C_1 + 2C_2 = 0$$

shift $n=2 \rightarrow k=1$, $n=k+1$

$$\Rightarrow \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k - \sum_{k=1}^{\infty} kC_k x^k - \sum_{k=1}^{\infty}(k+1)C_{k+1}x^k - \sum_{k=1}^{\infty} C_k x^k + 2C_2 - C_1 - C_0 = 0$$
$$\sum_{k=1}^{\infty}\left[(k+2)C_{k+2} - C_{k+1} - C_k\right](k+1)x^k + 2C_2 - C_1 - C_0 = 0$$

$$\begin{cases} 2C_2 = C_1 + C_0 \\ (k+2)C_{k+2} - C_{k+1} - C_k = 0, \quad k=1,2,3,\cdots \end{cases}$$

$C_2 = \frac{1}{2}C_0 + \frac{1}{2}C_1$

$k=1,\quad 3C_3 - C_2 - C_1 = 0,\quad \Rightarrow C_3 = \frac{1}{3}C_2 + \frac{1}{3}C_1 = \frac{1}{6}C_0 + \frac{1}{2}C_1$

$k=2,\quad 4C_4 = C_3 + C_2,\quad C_4 = \frac{1}{6}C_0 + \frac{1}{4}C_1$

$\vdots$

$y = C_0X^0 + C_1X + C_2X^2 + C_3X^3 + C_4X^4 + \cdots$

$= C_0 + C_1X + \frac{1}{2}C_0X^2 + \frac{1}{2}C_1X^2 + \frac{1}{6}C_0X^3 + \frac{1}{2}C_1X^3 + \frac{1}{6}C_0X^4 + \frac{1}{4}C_1X^4 + \cdots$

$= C_0\underbrace{(1 + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{6}X^4 + \cdots)}_{y_1} + C_1\underbrace{(X + \frac{1}{2}X^2 + \frac{1}{2}X^3 + \frac{1}{4}X^4 + \cdots)}_{y_2}$

5. Solution. $y' + 6y = e^{4t},\ y(0) = 2$

(1) $\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{e^{4t}\} \Rightarrow sY(s) - y(0) + 6Y(s) = \frac{1}{s-4}$

(2) $Y(s) = (\frac{1}{s-4} + 2)/(s+6) = \frac{1}{s-4}\cdot\frac{1}{s+6} + \frac{2}{s+6}$

$= \frac{1}{10}\left[\frac{1}{s-4} - \frac{1}{s+6}\right] + \frac{2}{s+6} = \frac{1}{10}\frac{1}{s-4} + \frac{19}{10}\frac{1}{s+6}$

(3) $\mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{1}{10}\mathcal{L}^{-1}\{\frac{1}{s-4}\} + \frac{19}{10}\mathcal{L}^{-1}\{\frac{1}{s+6}\} = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}$

6. Solution. $y'' + 5y' + 4y = 0,\ y(0) = 1,\ y'(0) = 0$ (1) $\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$

$s^2Y(s) - sy(0) - y'(0) + 5[sY(s) - y(0)] + 4Y(s) = 0,$

$[s^2 + 5s + 4]Y(s) = s + 5$; (2) $Y(s) = \frac{s+5}{s^2+5s+4} = \frac{A}{s+1} + \frac{B}{s+4}$

$s=-1,\ A = \frac{s+5}{s+4}\Big|_{s=-1} = \frac{4}{3}$; $s = -4,\ B = \frac{s+5}{s+1}\Big|_{s=-4} = -\frac{1}{3}$

(3) $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{4}{3}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s+4}\} = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$

7. Solution. $y'' - 4y' = 6e^{3t} - 3e^{-t},\ y(0) = 1,\ y'(0) = -1$

(1) $\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} = 6\mathcal{L}\{e^{3t}\} - 3\mathcal{L}\{e^{-t}\}; \Rightarrow s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] = \frac{6}{s-3} - 3\frac{1}{s+1}$

$s^2Y(s) - s + 1 - 4sY(s) + 4 = 6\cdot\frac{1}{s-3} - 3\cdot\frac{1}{s+1}$; $(s^2 - 4s)Y(s) = s - 5 + 6\cdot\frac{1}{s-3} - 3\cdot\frac{1}{s+1}$

(2) $Y(s) = \frac{s-5}{s^2-4s} + 6\cdot\frac{1}{s-3}\cdot\frac{1}{s^2-4s} - 3\frac{1}{s+1}\cdot\frac{1}{s^2-4s}$

$= \frac{5}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{1}{s-4} + 6\cdot\frac{1}{s}\cdot\left[\frac{1}{s-4} - \frac{1}{s-3}\right] - 3\cdot\frac{1}{s}\cdot\left[\frac{1}{s+1}\cdot\frac{1}{s-4}\right]$

$= \frac{5}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{1}{s-4} + 6\cdot\frac{1}{s}\cdot\frac{1}{s-4} - 6\cdot\frac{1}{s}\cdot\frac{1}{s-3} - \frac{3}{5}\cdot\frac{1}{s}\left[\frac{1}{s-4} - \frac{1}{s+1}\right]$

$= \frac{5}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{1}{s-4} + \frac{27}{5}\cdot\frac{1}{s}\cdot\frac{1}{s-4} - 6\cdot\frac{1}{s}\cdot\frac{1}{s-3} + \frac{3}{5}\frac{1}{s}\cdot\frac{1}{s+1} = \frac{5}{2}\frac{1}{s} + \frac{22}{20}\frac{1}{s-4} - 2\cdot\frac{1}{s-3} - \frac{3}{5}\frac{1}{s+1}$

$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{5}{2} + \frac{11}{10}e^{4t} - 2e^{3t} - \frac{3}{5}e^{-t}$