MATH 527
SPRING 2015
HW #7 SOLNS.
Variation of Parameters and LI.

1) $2y'' - 3y' + y = (t^2+1)e^{2t}$

constant coefficients.
2nd order problem
non-homog.

ansatz: $y_c = e^{\lambda t}$

$$2\lambda^2 - 3\lambda + 1 = 0$$

$$(2\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = \tfrac{1}{2}, 1 \qquad f(t) = \frac{(t^2+1)e^{2t}}{2}$$

$$\boxed{\begin{aligned} y_c(t) &= c_1 e^{\frac{1}{2}t} + c_2 e^t \\ y_p(t) &= u_1(t) e^{\frac{1}{2}t} + u_2(t) e^t \end{aligned}}$$

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{\frac{1}{2}t} & e^t \\ \frac{1}{2}e^{\frac{1}{2}t} & e^t \end{vmatrix}$$

$$W = e^{\frac{1}{2}t + t} - \tfrac{1}{2} e^{\frac{1}{2}t + t}$$

$$\boxed{W = \tfrac{1}{2} e^{\frac{1}{2}t + t} = \tfrac{1}{2} e^{\frac{3}{2}t}}$$

$$u_1'(t) = \frac{-y_2(t) f(t)}{W}$$

$$u_1'(t) = \frac{-e^t (t^2+1) e^{2t} \left(\frac{1}{2}\right)}{\frac{1}{2} e^{\frac{3}{2}t}}$$

$$u_1'(t) = -\frac{2(t^2+1)e^{\frac{3}{2}t}}{2}$$

$$u_1(t) = -\int (t^2+1) e^{\frac{3}{2}t}\, dt$$

IBP $u = t^2+1$ $\quad v = \frac{2}{3} e^{3/2 t}$

$du = 2t\,dt$ $\quad dv = e^{3/2\, t}\, dt$

$$u_1(t) = -\left((t^2+1)\tfrac{2}{3} e^{\frac{3}{2}t} - \int \tfrac{2}{3} e^{\frac{3}{2}t} 2t\, dt \right)$$

$$-\tfrac{4}{3} \int t e^{\frac{3}{2}t}\, dt$$

IBP: $u = t$ $\quad v = \frac{2}{3} e^{\frac{3}{2}t}$

$du = dt$ $\quad dv = e^{3/2 t}\, dt$

$$-\left[\tfrac{2}{3}(t^2+1) e^{\frac{3}{2}t} - \tfrac{4}{3}\left(t \tfrac{2}{3} e^{\frac{3}{2}t} - \cancel{\int \tfrac{2}{3} e^{\frac{3}{2}t} dt} \right) \right]$$

$$-\tfrac{4}{9} e^{\frac{3}{2}t}$$

$$\boxed{u_1(t) = -\tfrac{2}{3}(t^2+1) e^{\frac{3}{2}t} + \tfrac{8}{9} t e^{\frac{3}{2}t} - \tfrac{16}{27} e^{\frac{3}{2}t}}$$

$$u_2'(t) = \frac{y_1(t) f(t)}{W} = \frac{e^{\frac{1}{2}t}(t^2+1) e^{2t} \left(\frac{1}{2}\right)}{\frac{1}{2} e^{\frac{3}{2}t}}$$

$$u_2'(t) = (t^2+1) e^t$$

$$u_2(t) = \int (t^2+1) e^t\, dt$$

IBP $u = t^2+1$ $\quad v = e^t$

$du = 2t\,dt$ $\quad dv = e^t\, dt$

$$u_2(t) = \left((t^2+1) e^t - \int e^t 2t\, dt \right)$$

$$u_2(t) = \left[(t^2+1)e^t - 2\int te^t\,dt\right]$$

IBP $u = t$ $v = e^t$

$du = dt$ $dv = e^t\,dt$

$$-\left(te^t - \int e^t\,dt\right)$$

$$-e^t$$

$$u_2(t) = \left[(t^2+1)e^t - 2te^t + 2e^t\right]$$

$$y_p(t) = \left(-\frac{2}{3}(t^2+1)e^{\frac{3}{2}t} + \frac{8}{9}te^{\frac{3}{2}t} - \frac{16}{27}e^{\frac{3}{2}t}\right)e^{\frac{1}{2}t}$$

$$+\left((t^2+1)e^t - 2te^t + 2e^t\right)e^t$$

$$y_p(t) = \left(-\frac{2}{3}(t^2+1) + \frac{8}{9}t - \frac{16}{27}\right)e^{2t}$$

$$+\left((t^2+1) - 2t + 2\right)e^{2t}$$

$$y_p(t) = \left[\left(-\frac{2}{3} + \frac{3}{3}\right)(t^2+1) + \left(\frac{8}{9} - \frac{18}{9}\right)t\right.$$

$$\left. + \left(-\frac{2}{3} - \frac{16}{27} + 1 + 2\right)\right]e^{2t}$$

$$y_p(t) = \left[\frac{1}{3}(t^2+1) - \frac{10}{9}t\right.$$

$$\left. + \left(-\frac{18}{27} - \frac{16}{27} + \frac{27}{27} + \frac{54}{27}\right)\right]e^{2t}$$

$$y_p(t) = \left[\frac{1}{3}(t^2+1) - \frac{10}{9}t + \frac{47}{27}\right]e^{2t}$$

$$\boxed{y_p(t) = \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}}$$

$$\boxed{y_{gen'l}(t) = c_1 e^{\frac{1}{2}t} + c_2 e^t + \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}}$$

2) $y'' + y = \sec(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$y_c(t) = e^{\lambda t}$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_c(t) = c_1 e^{it} + c_2 e^{-it}$ OR

$y_c(t) = \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t)$

$y_p(t) = u_1(t)\cos(t) + u_2(t)\sin(t)$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$

$W = \cos^2 t + \sin^2 t = 1$

$u_1'(t) = \frac{-y_2(t) f(t)}{W}$

$= \frac{-\sin(t)\sec(t)}{1}$

$u_1(t) = -\int \frac{\sin(t)}{\cos(t)} dt$

$=$ $u = \cos(t)$
$du = -\sin(t)\,dt$

$u_1(t) = \int \frac{1}{u} dt = \ln|\cos(t)| + \cancel{c}^{\,0}$

$u_1(t) = \ln|\cos(t)|$, $\cos(t) > 0$ for

$\boxed{u_1(t) = \ln(\cos(t))}$ $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$u_2'(t) = \frac{y_1(t) f(t)}{W} = \frac{\cos(t)\sec(t)}{1}$

$u_2'(t) = 1$

$u_2(t) = \int 1\, dt$

$u_2(t) = t + \cancel{c}^{\,0}$

$\boxed{u_2(t) = t}$

$\boxed{y_p(t) = \ln(\cos(t))\cos(t) + t\sin(t)}$

$$\boxed{\begin{aligned} y_{gen'l}(t) &= \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t) \\ &+ \ln(\cos(t))\cos(t) + t\sin(t) \end{aligned}}$$

3) $y'' - 3y' + 2y = te^{3t} + 1$

$y_c(t) = e^{\lambda t}$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1, 2$

$$\boxed{\begin{aligned} y_c(t) &= c_1 e^t + c_2 e^{2t} \\ y_p(t) &= u_1 e^t + u_2 e^{2t} \end{aligned}}$$

$$W = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = 2e^{3t} - e^{3t}$$

$$\boxed{W = e^{3t}}$$

$$u_1'(t) = \frac{-y_2(t) f(t)}{W}$$

$$= \frac{-e^{2t}(te^{3t} + 1)}{e^{3t}}$$

$$= -(te^{3t} + 1)e^{-t}$$

$$u_1'(t) = -te^{2t} - e^{-t}$$

$$u_1(t) = -\int te^{2t}\,dt - \int e^{-t}\,dt$$

IBP $\quad u = t \quad v = \frac{1}{2}e^{2t}$
$\quad du = dt \quad dv = e^{2t}\,dt$

$$-\left(\tfrac{1}{2}te^{2t} - \int \tfrac{1}{2}e^{2t}\,dt\right) - \int e^{-t}\,dt$$

$$u_1 = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t} + \cancel{C}^0$$

$$\boxed{u_1(t) = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}}$$

$$u_2'(t) = \frac{y_1(t) f(t)}{W} = e^t(te^{3t} + 1)e^{-3t}$$

$$= (te^{4t} + e^t)e^{-3t}$$

$$= te^t + e^{-2t}$$

$$u_2(t) = \int te^t\,dt + \int e^{-2t}\,dt$$

$$u_2(t) = te^t - e^t + -\frac{1}{2}e^{-2t} + \cancel{C}^0$$

$$\boxed{u_2(t) = te^t - e^t - \frac{1}{2}e^{-2t}}$$

$$y_p(t) = \left(-\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}\right)e^t$$

$$+ \left(te^t - e^t - \frac{1}{2}e^{-2t}\right)e^{2t}$$

$$y_p(t) = -\frac{1}{2}te^{3t} + \frac{1}{4}e^{3t} + e^0$$

$$+ te^{3t} - e^{3t} - \frac{1}{2}e^0$$

$$\boxed{y_p(t) = \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \cancel{e^{2t}}^{\,\text{LD}} + \frac{1}{2}}$$

$$\boxed{y_{gen'l}(t) = c_1 e^t + c_2 e^{2t} + \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \frac{1}{2}}$$

4) $3y'' + 4y' + y = e^{-t}\sin t,\quad y(0)=1,\ y'(0)=0$

$$y_c(t) = e^{\lambda t}$$

$$\boxed{f(t) = \frac{e^{-t}\sin t}{3}}$$

$$3\lambda^2 + 4\lambda + 1 = 0$$

$$(3\lambda+1)(\lambda+1) = 0$$

$$\lambda = -\frac{1}{3}, -1$$

$$\boxed{\begin{aligned} y_c(t) &= c_1 e^{-\frac{1}{3}t} + c_2 e^{-t} \\ y_p(t) &= u_1(t) e^{-\frac{1}{3}t} + u_2(t) e^{-t} \end{aligned}}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{3}t} & e^{-t} \\ -\frac{1}{3}e^{-\frac{1}{3}t} & -e^{-t} \end{vmatrix}$$

$$W = -e^{-\frac{1}{3}t - t} + +\frac{1}{3}e^{-\frac{1}{3}t - t}$$

$$\boxed{W = -\frac{2}{3}e^{-\frac{4}{3}t}}$$

$$u_1'(t) = \frac{-y_2(t) f(t)}{W} = \frac{-e^{-t}\frac{e^{-t}\sin(t)}{3}}{-\frac{2}{3}e^{-\frac{4}{3}t}}$$

$$u_1'(t) = \frac{1}{3}\frac{3}{2}\sin(t)\, e^{-2t + \frac{4}{3}t}$$

$$u_1'(t) = \frac{1}{2}\sin(t) e^{-\frac{2}{3}t}$$

$$u_1(t) = \frac{1}{2}\int \sin(t) e^{-\frac{2}{3}t}\,dt$$

$u = \sin(t) \quad v = -\frac{3}{2}e^{-\frac{2}{3}t}$

$du = \cos(t)\,dt \quad dv = e^{-\frac{2}{3}t}\,dt$

$$= \frac{1}{2}\left(-\frac{3}{2}\sin(t)e^{-\frac{2}{3}t} - \int -\frac{3}{2}e^{-\frac{2}{3}t}\cos(t)\,dt\right)$$

$u = \cos(t) \quad v = -\frac{3}{2}e^{-\frac{2}{3}t}$

$du = -\sin(t)\,dt \quad dv = e^{-\frac{2}{3}t}\,dt$

$$u_1(t) = \frac{1}{2}\left(-\frac{3}{2}\sin(t)e^{-\frac{2}{3}t} + \frac{3}{2}\left(\cos(t)\left(-\frac{3}{2}e^{-\frac{2}{3}t}\right) - \int +\frac{3}{2}e^{-\frac{2}{3}t}\sin t\,dt\right)\right)$$

Let $I = \int \sin(t) e^{-\frac{2}{3}t}\,dt$

$$u_1(t) = \frac{1}{2}I$$

$$\frac{1}{2}I = \frac{1}{2}\left(-\frac{3}{2}\sin(t)e^{-\frac{2}{3}t} - \frac{9}{4}\cos(t)e^{-\frac{2}{3}t} - \frac{9}{4}I\right)$$

$$\left(\frac{4}{4} + \frac{9}{4}\right)I = \frac{3}{2}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}$$

$$I = \frac{4}{13}\cdot\frac{3}{2}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}$$

$$I = \frac{6}{13}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t} + \cancel{C}^{0}$$

$$\boxed{u_1(t) = \frac{1}{2}I = \frac{3}{13}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}}$$

$$u_2'(t) = \frac{y_1(t) f(t)}{W} = \frac{e^{-\frac{1}{3}t}\left(e^{-t}\sin(t)\frac{1}{3}\right)}{-\frac{2}{3}e^{-\frac{4}{3}t}}$$

$$= -\frac{1}{3}\frac{3}{2}\sin(t)e^{-\frac{1}{3}t - t + \frac{4}{3}t}$$

$$u_2'(t) = -\frac{1}{2}\sin(t)e^{0t}$$

$$u_2(t) = -\frac{1}{2}\int \sin(t)\,dt$$

$$u_2(t) = -\frac{1}{2}\left(-\cos(t) + \cancel{C}^{0}\right)$$

$$\boxed{u_2(t) = \frac{1}{2}\cos(t)}$$

$$y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+\left(\frac{3}{13}\left(\frac{3}{2}\cos(t) - \sin(t)\right) e^{-\frac{2}{3}t}\right) e^{-\frac{1}{3}t}$$

$$+\left(\frac{1}{2}\cos(t)\right)e^{-t}$$

$$y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+\frac{3}{13}\left(\frac{3}{2}\cos(t) - \sin(t)\right)e^{-t}$$

$$+\left(\frac{1}{2}\cos(t)\right)e^{-t}$$

$$\boxed{y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t} + \left[\left(-\frac{9}{26} + \frac{13}{26}\right)\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t}}$$

$$\boxed{y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t} + \left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t}}$$

1c) $y(0) = 1$, $y'(0) = 0$

$y(0) = 1$

$$1 = c_1 + c_2 + \frac{2}{13}$$

$$\boxed{c_1 = \frac{11}{13} - c_2}$$

$y'(0) = 0$

$$\boxed{y'(t) = c_1\left(-\frac{1}{3}\right)e^{-\frac{1}{3}t} - c_2 e^{-t} - \left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t} + e^{-t}\left(-\frac{2}{13}\sin(t) - \frac{3}{13}\cos(t)\right)}$$

$y'(0) = 0$

$$0 = -\frac{1}{3}c_1 - c_2 - \frac{2}{13} - \frac{3}{13}$$

$$0 = -\frac{1}{3}\left(\frac{11}{13} - c_2\right) - c_2 - \frac{5}{13}$$

$$\frac{5}{13} + \frac{11}{39} = \left(+\frac{1}{3} - 1\right)c_2$$

$$\frac{15}{39} + \frac{11}{39}$$

$$\frac{26}{39}$$

$$\frac{2}{3} = -\frac{2}{3}c_2$$

$$\boxed{c_2 = -1} \rightarrow c_1 = \frac{11}{13} - \left(-\frac{13}{13}\right)$$

$$\boxed{c_1 = \frac{24}{13}}$$

$$y(t) = \frac{24}{13}e^{-\frac{1}{3}t} - e^{-t} + \left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t}$$

$$\boxed{y(t) = \frac{1}{13}e^{-t}\left(24e^{\frac{2}{3}t} - 13 + 2\cos(t) - 3\sin(t)\right)}$$

Problem 5.

a) Short answer: For an operator L to be linear, it must be true that

$$L(c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)) = c_1 L(y_1(x)) + \dots + c_n L(y_n(x))$$

for $c_1, c_2, \dots, c_n$ real-valued numbers and $y_1(x), y_2(x), \dots y_n(x)$ functions in the domain of L.

Extended answer: That equation means that you can apply the operator L (e.g., a derivative) to the linear combination of the functions $(y_1(x), y_2(x), \dots, y_n(x))$ and the result is the same as if you applied the operator to each of the functions separately and then took the linear combination.

b) For functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ to be linearly dependent, at least one of the functions can be written as a linear combination of the other functions.

Formally, for the equation $c_1 y_1(x) + \ldots + c_n y_n(x) = 0$ not all constants $c_1, c_2, \ldots, c_n$ are equal to zero.

c) For functions $\{y_1(x), y_2(x), \ldots y_n(x)\}$ to be linearly independent, none of the functions can be written as linear combination of the other functions.

Formally, for the equation $c_1 y_1(x) + \ldots + c_n y_n(x) = 0$, all constants $c_1, c_2, \ldots, c_n$ are equal to zero.

d) Linearity allows us to form gen'l solns from a linear combination of linearly independent solns.

The general solution to a linear differential eqn is a linear combination of the Linearly independent solutions and each linearly independent term is a set of linearly dependent solutions.

$\therefore$ These properties are properties of the general solution.

e) An n^{th} order linear homogeneous equation has n linearly independent solns.

f) A $y_p(x)$ to a nonhomogeneous equation is linearly independent to $y_c(x)$.

g) $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

h) To prove Euler's formula, I would express the functions on both right- and left-hand sides of the equation as Taylor Expansions about $x=0$. Then I can show that LHS is equal to RHS.