Math 527, Spring 2015
Homework 6 Solutions

Problems 1-6. Find the general solution of the differential equation using the method of judicious guessing.

1. $y'' + y' - 6y = 2x$

Solution. First we find the complementary solution y_c, i.e., the solution to the associated homogeneous equation $y'' + y' - 6y = 0$:

$$\begin{aligned} \lambda^2 + \lambda - 6 &= 0 \\ (\lambda+3)(\lambda-2) &= 0 \\ \lambda_1 = -3, \;\; \lambda_2 &= 2 \qquad \Longrightarrow \qquad y_c = C_1 e^{-3x} + C_2 e^{2x}. \end{aligned}$$

Next we find the particular solution y_p. We guess

$$\begin{aligned} & y_p = Ax + B \\ \Longrightarrow \quad & y_p' = A \\ \Longrightarrow \quad & y_p'' = 0. \end{aligned}$$

Plugging y_p into the ODE and rearranging terms gives

$$-6Ax + (A - 6B) = 2x,$$

which yields equations

$$\begin{aligned} -6A &= 2 \\ A - 6B &= 0. \end{aligned}$$

Solving this system gives

$$A = -\frac{1}{3} \qquad B = -\frac{1}{18}.$$

Therefore

$$y_p = -\frac{x}{3} - \frac{1}{18}$$

and the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^{-3x} + C_2 e^{2x} - \frac{x}{3} - \frac{1}{18}. \end{aligned}$$ □

2. $y'' - 10y' + 25y = 30x + 3$

Solution. First we find the complementary solution y_c:

$$\begin{aligned} \lambda^2 - 10\lambda + 25 &= 0 \\ (\lambda - 5)^2 &= 0 \\ \lambda_1 = \lambda_2 &= 5 \qquad \Longrightarrow \qquad y_c = C_1 e^{5x} + C_2 x e^{5x}. \end{aligned}$$

Next we find the particular solution y_p. We guess

$$\begin{aligned} & y_p = Ax + B \\ \Longrightarrow \quad & y_p' = A \\ \Longrightarrow \quad & y_p'' = 0. \end{aligned}$$

Plugging y_p into the ODE and rearranging terms gives

$$25Ax + (-10A + 25B) = 30x + 3,$$

while yields equations

$$\begin{aligned} 25A &= 30 \\ -10A + 25B &= 3. \end{aligned}$$

Solving this system gives

$$A = \frac{6}{5} \qquad\qquad B = \frac{3}{5}.$$

Therefore

$$y_p = \frac{6x}{5} + \frac{3}{5}$$

and the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^{5x} + C_2 x e^{5x} + \frac{6x}{5} + \frac{3}{5}. \end{aligned}$$

□

3. $4y'' - 4y' - 3y = \cos(2x)$

Solution. First we find the complementary solution y_c:

$$\begin{aligned} 4\lambda^2 - 4\lambda - 3 &= 0 \\ (2\lambda + 1)(2\lambda - 3) &= 0 \\ \lambda_1 = -\frac{1}{2}, \ \ \lambda_2 = \frac{3}{2} \qquad &\Longrightarrow \qquad y_c = C_1 e^{-x/2} + C_2 e^{3x/2}. \end{aligned}$$

Next we find the particular solution y_p. We guess

$$\begin{aligned} y_p &= A\cos(2x) + B\sin(2x) \\ \Longrightarrow \quad y_p' &= -2A\sin(2x) + 2B\cos(2x) \\ \Longrightarrow \quad y_p'' &= -4A\cos(2x) - 4B\sin(2x). \end{aligned}$$

Plugging y_p into the ODE and simplifying gives

$$(-19A - 8B)\cos(2x) + (8A - 19B)\sin(2x) = \cos(2x),$$

which yields the equations

$$\begin{aligned} -19A - 8B &= 1 \\ 8A - 19B &= 0. \end{aligned}$$

Solving this system gives

$$A = -\frac{19}{425} \qquad\qquad B = -\frac{8}{425}.$$

Therefore

$$y_p = -\frac{19}{425}\cos(2x) - \frac{8}{425}\sin(2x)$$

and the general solution is

$$\begin{aligned} y &= y_p + y_c \\ &= C_1 e^{-x/2} + C_2 e^{3x/2} - \frac{19}{425}\cos(2x) - \frac{8}{425}\sin(2x). \end{aligned}$$

□

4. $y'' - y' + y/4 = 3 + e^{x/2}$

Solution. To eliminate fractions, we multiply both sides of the equation by 4:

$$4y'' - 4y' + y = 12 + 4e^{x/2}.$$

We then find the complementary solution y_c:

$$\begin{aligned} 4\lambda^2 - 4\lambda + 1 &= 0 \\ (2\lambda - 1)^2 &= 0 \\ \lambda_1 = \lambda_2 &= \frac{1}{2} \qquad \Longrightarrow \qquad y_c = C_1 e^{x/2} + C_2 x e^{x/2}. \end{aligned}$$

Since the RHS of the ODE is the sum of functions from two different families, we find the particular solution in two steps. First we find a solution y_{p_1} to the equation

$$4y'' - 4y' + y = 12. \tag{1}$$

We guess

$$y_{p_1} = A \quad \Longrightarrow \quad y'_{p_1} = y''_{p_1} = 0.$$

Plugging y_{p_1} into (1) gives $A = 12$, hence

$$y_{p_1} = 12.$$

Next we find a solution y_{p_2} to the equation

$$4y'' - 4y' + y = e^{x/2}. \tag{2}$$

The obvious first guess, $y_{p_2} = Be^{x/2}$, won't work because it duplicates a term in y_c. Our second guess, $y_{p_2} = Bxe^{x/2}$, runs into the same issue. So we use

$$\begin{aligned} y_{p_2} &= Bx^2 e^{x/2} \\ \Longrightarrow \quad y'_{p_2} &= 2Bxe^{x/2} + \frac{1}{2}Bx^2 e^{x/2}, \\ \Longrightarrow \quad y''_{p_2} &= 2Be^{x/2} + 2Bxe^{x/2} + \frac{1}{4}Bx^2 e^{x/2}. \end{aligned}$$

Plugging y_{p_2} into (2) and simplifying gives $B = 1/2$, hence

$$y_{p_2} = \frac{1}{2}x^2 e^{x/2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 e^{x/2} + C_2 x e^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}. \end{aligned}$$ □

5. $y'' - 16y = 2e^{4x}$.

Solution. First we find the complementary solution y_c:

$$\begin{aligned} \lambda^2 - 16 &= 0 \\ (\lambda+4)(\lambda-4) &= 0 \\ \lambda_1 = 4, \quad \lambda_2 &= -4 \qquad \implies \qquad y_c = C_1 e^{4x} + C_2 e^{-4x}. \end{aligned}$$

Next we find the particular solution y_p. Our first guess, $y_p = Ae^{4x}$, won't work because it duplicates a term in y_c. So instead we guess

$$\begin{aligned} y_p &= Axe^{4x} \\ \implies \quad y_p' &= Ae^{4x} + 4Axe^{4x} \\ \implies \quad y_p'' &= 8Ae^{4x} + 16Axe^{4x}. \end{aligned}$$

Plugging y_p into the ODE and simplifying gives $A = 1/4$, hence

$$y_p = \frac{1}{4}xe^{4x}.$$

Therefore the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^{4x} + C_2 e^{-4x} + \frac{1}{4}xe^{4x}. \end{aligned}$$

□

6. $y'' - 2y' + 5y = e^x \cos(2x)$

Solution. First we find the complementary solution y_c:

$$\begin{aligned} \lambda^2 - 2\lambda + 5 &= 0 \\ \lambda &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \qquad \Longrightarrow \qquad y_c = C_1 e^x \cos(2x) + C_2 e^x \sin(2x). \end{aligned}$$

Next we find the particular solution y_p. The obvious first guess, $y_p = e^x(A\cos(2x) + B\sin(2x))$, won't work because it duplicates a term in y_c. So instead we guess

$$y_p = xe^x(A\cos(2x) + B\sin(2x)).$$

To simplify calculations, let $s = A\cos(2x) + B\sin(2x)$ so that $y_p = xe^x s$. Then $s'' = -4s$ and we get

$$\begin{aligned} y_p' &= e^x s + xe^x s + xe^x s' \\ y_p'' &= 2e^x s + 2e^x s' - 3xe^x s + 2xe^x s'. \end{aligned}$$

Plugging y_p into the ODE and simplifying gives

$$\begin{aligned} 2e^x s' &= e^x \cos(2x) \\ 2s' &= \cos(2x) \\ -4A\sin(2x) + 4B\cos(2x) &= \cos(2x). \end{aligned}$$

This gives $A = 0$ and $B = 1/4$, hence

$$y_p = \frac{1}{4} xe^x \sin(2x).$$

The general solution is then

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^x \cos(2x) + C_2 e^x \sin(2x) + \frac{1}{4} xe^x \sin(2x). \end{aligned}$$ □

Problems 7 and 8. Solve the initial value problem.

7. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \qquad y(0) = 2, \quad y'(0) = 5.$

Solution. First we find the complementary solution y_c:

$$\begin{aligned} \lambda^2 + 4\lambda + 4 &= 0 \\ (\lambda + 2)^2 &= 0 \\ \lambda_1 = \lambda_2 &= -2 \qquad \Longrightarrow \qquad y_c = C_1 e^{-2x} + C_2 x e^{-2x}. \end{aligned}$$

Next we find the particular solution y_p. Our first guess, $y_p = (Ax + B)e^{-2x}$, won't work because it duplicates a term in y_c. So we multiply this guess by powers of x until we eliminate duplicates; this requires

$$\begin{aligned} y_p &= x^2(Ax + B)e^{-2x} = (Ax^3 + Bx^2)e^{-2x} \\ \Longrightarrow \quad y_p' &= -2y_p + (3Ax^2 + 2Bx)e^{-2x} \\ \Longrightarrow \quad y_p'' &= 4y_p - 4(3Ax^2 + 2Bx)e^{-2x} + (6Ax + 2B)e^{-2x}. \end{aligned}$$

Plugging y_p into the ODE and simplifying gives

$$6Ax + 2B = 3 + x.$$

Solving this system gives $A = 1/6$, $B = 3/2$, hence

$$y_p = \left(\frac{x^3}{6} + \frac{3x^2}{2} \right) e^{-2x}.$$

Therefore the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^{-2x} + C_2 x e^{-2x} + \left(\frac{x^3}{6} + \frac{3x^2}{2} \right) e^{-2x}. \end{aligned}$$

The value $y(0) = 2$ gives $C_1 = 2$, while the value $y'(0) = 5$ gives $C_2 = 9$. Thus the solution to the IVP is

$$y = 2e^{-2x} + 9xe^{-2x} + \left(\frac{x^3}{6} + \frac{3x^2}{2} \right) e^{-2x}.$$ □

8. $y'' - y = \cosh x, \qquad y(0) = 2, \quad y'(0) = 12.$

Solution. Note that $\cosh x = e^x/2 + e^{-x}/2$, so we can rewrite the ODE as

$$y'' - y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

First we find the complementary solution y_c:

$$\begin{aligned} \lambda^2 - 1 &= 0 \\ \lambda^2 &= 1 \\ \lambda_1 = 1, \ \ \lambda_2 &= -1 \qquad \implies \qquad y_c = C_1 e^x + C_2 e^{-x}. \end{aligned}$$

We find the particular solution in two steps. First, we find a solution y_{p_1} to the equation

$$y'' - y = \frac{1}{2}e^x. \tag{1}$$

Our first guess, $y_{p_1} = Ae^x$, won't work because it duplicates a term in y_c. So instead we guess

$$\begin{aligned} y_{p_1} &= Axe^x \\ \implies \quad y'_{p_1} &= y_{p_1} + Ae^x \\ \implies \quad y''_{p_1} &= y_{p_1} + 2Ae^x. \end{aligned}$$

Plugging y_{p_1} into (1) and simplifying gives

$$2A = \frac{1}{2} \quad \implies \quad A = \frac{1}{4}. \quad \implies \quad y_{p_1} = \frac{1}{4}xe^x.$$

Next we find a solution y_{p_2} to the equation

$$y'' - y = \frac{1}{2}e^{-x}. \tag{2}$$

Our first guess, $y_{p_2} = Be{-}x$, won't work because it duplicates a term in y_c. So instead we guess

$$\begin{aligned} y_{p_2} &= Bxe^{-x} \\ \implies \quad y'_{p_2} &= -y_{p_2} + Be^{-x} \\ \implies \quad y''_{p_2} &= y_{p_2} - 2Be^{-x}. \end{aligned}$$

Plugging y_{p_2} into (2) and simplifying gives

$$-2B = \frac{1}{2} \quad \implies \quad B = -\frac{1}{4} \quad \implies \quad y_{p_2} = -\frac{1}{4}xe^{-x}.$$

Therefore the general solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 e^x + C_2 e^{-x} + \frac{1}{4}xe^x - \frac{1}{4}xe^{-x}. \end{aligned}$$

Plugging in the initial values gives us a system of equations

$$\begin{aligned} C_1 + C_2 &= 2 \\ C_1 - C_2 &= 12. \end{aligned}$$

Solving this system gives $C_1 = 7$ and $C_2 = -5$. Therefore the solution to the IVP is

$$y = 7e^x - 5e^{-x} + \frac{1}{4}xe^x - \frac{1}{4}xe^{-x}.$$

□