MATH 527 Solutions of HW#5

1. Solutions $y''-3y'+y=0$; ① Assume $y=e^{\lambda t}$, then $y'=\lambda e^{\lambda t}$, $y''=\lambda^2 e^{\lambda t}$

$\Rightarrow e^{\lambda t}(\lambda^2-3\lambda+1)=0$ —— auxiliary equation $\lambda^2-3\lambda+1=0$ ②

③ $\lambda=\frac{3\pm\sqrt{9-4\cdot1\cdot1}}{2\cdot1}=\frac{3\pm\sqrt{5}}{2}$ $\lambda_1=\frac{3}{2}+\frac{\sqrt5}{2}$; $\lambda_2=\frac{3}{2}-\frac{\sqrt5}{2}$, (case 1: real, distinct)

④ So the general solution is $y=C_1e^{(\frac32+\frac{\sqrt5}{2})t}+C_2e^{(\frac32-\frac{\sqrt5}{2})t}$.

2. $2y''+3y'+4y=0$. Repeat the similar steps, $\Rightarrow 2\lambda^2+3\lambda+4=0$

$\lambda=\frac{-3\pm\sqrt{9-4\cdot2\cdot4}}{2\cdot2}=\frac{-3\pm\sqrt{-23}}{4}=-\frac34\pm i\sqrt{23}$ (Case 3, complex)

general solution $y=C_1e^{(-\frac34+i\sqrt{23})t}+C_2e^{(-\frac34-i\sqrt{23})t}$, using Euler's formular

$\Rightarrow y=(\tilde{C}_1\cos\sqrt{23}t+\tilde{C}_2\sin\sqrt{23}t)e^{-\frac34 t}$

3. $4y''-12y'+9y=0 \xRightarrow{y=e^{\lambda t}} 4\lambda^2-12\lambda+9=0$ $(2\lambda-3)^2=0$, $\lambda_1=\lambda_2=\frac32$

(Case 2, real, repeat) general solution $y=C_1e^{\frac32t}+C_2te^{\frac32t}$

4. $9y''+6y'+y=0$; $y(0)=1$, $y'(0)=0 \xRightarrow{y=e^{\lambda t}} 9\lambda^2+6\lambda+1=0$

(Don't touch I.C until you find the general solution) $(3\lambda+1)^2=0$, $\lambda_1=\lambda_2=-\frac13$

general solution $y=C_1e^{-\frac13t}+C_2te^{-\frac13t}$, now we can use I.C

$y(0)=1$, $C_1+0=1\Rightarrow C_1=1$, $y'=-\frac13C_1e^{-\frac13t}+C_2e^{-\frac13t}-\frac13C_2te^{-\frac13t}$

$y'(0)=0\Rightarrow -\frac13+C_2=0$ $\therefore C_2=\frac13$, $y=e^{-\frac13t}+\frac13te^{-\frac13t}$

5. $5y''+5y'-y=0$; $y(0)=0$, $y'(0)=1\Rightarrow 5\lambda^2+5\lambda-1=0$, $\lambda=\frac{-5\pm\sqrt{25+4\cdot5\cdot1}}{2\cdot5}=\frac{-5\pm3\sqrt5}{10}$

(Case 1. real, distinct) $y=C_1e^{\frac{-5+3\sqrt5}{10}t}+C_2e^{\frac{-5-3\sqrt5}{10}t}$

$y'=C_1(\frac{-5+3\sqrt5}{10})e^{(\frac{-5+3\sqrt5}{10})t}+C_2(\frac{-5-3\sqrt5}{10})e^{\frac{-5-3\sqrt5}{10}t}$

plug in $y(0)=0$, $C_1+C_2=0$; $y'(0)=1$, $C_1\frac{-5+3\sqrt5}{10}+C_2\frac{-5-3\sqrt5}{10}=1$

$\Rightarrow C_1=\frac{\sqrt5}{3}$, $C_2=-\frac{\sqrt5}{3}$, $\therefore y=\frac{-5\sqrt5+15}{30}e^{\frac{-5+3\sqrt5}{10}t}+\frac{5\sqrt5+15}{30}e^{\frac{-5-3\sqrt5}{10}t}$

$=\frac{-\sqrt5+3}{6}e^{\frac{-5+3\sqrt5}{10}t}+\frac{\sqrt5+3}{6}e^{\frac{-5-3\sqrt5}{10}t}$

6. $\Rightarrow \lambda^2+2\lambda+5=0$, $(\lambda+1)^2+4=0$ $\therefore \lambda=-1\pm2i$, $y=e^{-t}(C_1\cos2t+C_2\sin2t)$

$y(0)=C_1=0$, then $y=e^{-t}\cdot C_2\sin2t$, $y'=-e^{-t}C_2\sin2t+e^{-t}\cdot C_2\cdot2\cos2t$

$y'(0)=2C_2=2$ $\therefore C_2=1$, $y=e^{-t}\sin2t$

7. proof. (a) $y_1 = e^{iwt}$, $y_1'' = -w^2 e^{iwt}$, $y_1'' + w^2 y_1 = 0$, $\therefore$ y_1 is a solution.
$y_2 = e^{-iwt}$, $y_2'' = -w^2 e^{-iwt}$, $y_2'' + w^2 y_2 = 0$, $\therefore$ y_2 is a solution

Assume $\alpha \neq 0$, $\beta \neq 0$, $\exists$ $\alpha y_1 + \beta y_2 = 0$ $\Rightarrow$ $y_1 = -\frac{\beta}{\alpha} y_2$, α, β are constants

$e^{iwt} = -\frac{\beta}{\alpha} e^{-iwt}$ X It's impossible. You can set $-\frac{\beta}{\alpha} = 1$, but the powers of e at two sides can't equal ! $\therefore$ $\alpha = 0$, $\beta = 0$, y_1, y_2 are linearly indept.

(b) $\hat{y}_1(t) = a_1 y_1(t) + a_2 y_2(t)$; $\hat{y}_2(t) = b_1 y_1(t) + b_2 y_2(t) = b_1 e^{iwt} + b_2 e^{-iwt}$

$= a_1 e^{iwt} + a_2 e^{-iwt}$;

$= \cos wt\,(a_1 + a_2) + i(a_1 - a_2)\sin wt$;

$= \cos wt$

$\therefore$ $a_1 + a_2 = 1$, $a_1 - a_2 = 0$ $\Rightarrow$ $a_1 = a_2 = \frac{1}{2}$;

$= \cos wt\,(b_1 + b_2) + i(b_1 - b_2)\sin wt$

$= \sin wt$

$b_1 + b_2 = 0$, $i(b_1 - b_2) = 1$

$b_1 = -\frac{i}{2}$, $b_2 = \frac{i}{2}$

(c) $\hat{y}_1'' = -w^2 \cos wt$, $\hat{y}_1'' + w^2 \hat{y}_1 = 0$ $\therefore$ $\hat{y}_1$ is a solution

$\hat{y}_2'' = -w^2 \sin wt$, $\hat{y}_2'' + w^2 \hat{y}_2 = 0$ $\therefore$ $\hat{y}_2$ is a solution.

Assume $\exists$ $\alpha \neq 0, \beta \neq 0$, $\alpha \hat{y}_1 + \beta \hat{y}_2 = 0$, α, β are constants

$\Rightarrow$ $\cos wt = -\frac{\beta}{\alpha} \sin wt$ X It's impossible. $\therefore$ $\alpha = 0 = \beta$. $\hat{y}_1, \hat{y}_2$ are L. Indp.

(d) It's homogenous D.E., $\therefore$ $y = C_1 \hat{y}_1(t) + C_2 \hat{y}_2(t) = C_1 \cos wt + C_2 \sin wt$

8. proof. $(\cos x + i \sin x)^n = (e^{ix})^n = e^{inx} = \cos nx + i \sin nx$

$n = 2$, $(\cos x + i \sin x)^2 = \cos^2 x - \sin^2 x + i \cdot 2 \sin x \cos x = \cos 2x + i \sin 2x$

$\therefore$ $\cos 2x = \cos^2 x - \sin^2 x$; $\sin 2x = 2 \sin x \cos x$

9. Using the ansatz $y(t) = e^{\lambda t}$ $\Rightarrow$ $[t\lambda^2 - (1+3t)\lambda + 3] e^{\lambda t} = 0$ $\therefore$ $t\lambda^2 - (1+3t)\lambda + 3 = 0$

$\lambda = 3$ or $\frac{1}{t}$. if $y = e^{3t}$, $\Rightarrow$ $[9t - (1+3t)3 + 3] e^{3t} = 0$ ✓

when $y = e^{\frac{1}{t} \cdot t} = e$ plug in DE., $3e = 0$ X ; so $y_1 = e^{3t}$, assume $y_2 = u(t) \cdot y_1$

transform the D.E. ~~into~~ $y'' - (3 + \frac{1}{t}) y' + \frac{3}{t} y = 0$, $p(t) = -(3 + \frac{1}{t})$

$u' = \frac{C_1 e^{-\int p\,dt}}{y_1^2} = C_1 e^{-3t} |t|$, $u = C_1 \int e^{-3t} |t|\,dt = -\frac{C}{3}[t e^{-3t} + \frac{1}{3} e^{-3t}]$

$y_2 = u \cdot y_1 = -\frac{C}{3}[t + \frac{1}{3}]$, it satisfies the original DE. $\therefore$ $y = C_1 e^{3t} + C_2 (t + \frac{1}{3})$

10. Using $y = t^\lambda$ $\Rightarrow$ $[\lambda(\lambda - 1) + 5\lambda - 5] t^\lambda = 0$ $\therefore$ $\lambda^2 + 4\lambda - 5 = 0$ $\lambda_1 = 1$, $\lambda_2 = -5$

$y_1 = t$, $y_2 = t^{-5}$, plug them back the D.E., both are the solutions ✓

Assume $\exists$ $\alpha \neq 0, \beta \neq 0$, $\alpha t + \beta t^{-5} = 0$ $\Rightarrow$ $t = -\frac{\beta}{\alpha} t^{-5}$ X (α, β are constants)

$\therefore$ $\alpha = 0 = \beta$, y_1, y_2 are linearly Indpt.