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MATH 527
SPRING 2015
HW #4 SOLNS

Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_m)$$

T_m = ambient temp = 10°F

$T(0) = 70$

$T(.5) = 50$

$$\frac{dT}{dt} = k(T-10)$$

$$\left(\frac{1}{T-10}\right)\frac{dT}{dt} = k$$

$$\frac{d}{dt}\left(\ln|T-10|\right) = k$$

$$\int \frac{d}{dt}\left(\ln|T-10|\right)dt = \int k\,dt$$

$$\ln|T-10| = kt + C$$

$$T - 10 = Ce^{kt}$$

$$T(t) = 10 + Ce^{kt}$$

$T(0) = 70$

$$T(0) = 10 + C = 70$$

$$\boxed{C = 60}$$

$$T(t) = 10 + 60e^{kt}$$

$T(.5) = 50$

Solve for k.

$$T(.5) = 10 + 60e^{k/2} = 50$$

$$60e^{k/2} = 40$$

$$e^{k/2} = \frac{2}{3}$$

$$\frac{k}{2} = \ln\left(\frac{2}{3}\right)$$

$$\boxed{k = 2\ln\left(\frac{2}{3}\right) = \ln\left(\frac{4}{9}\right)}$$

$$\boxed{T(t) = 10 + 60e^{\ln\left(\frac{4}{9}\right)t}}$$

1) $T(1) = ?$

$$T(1) = 10 + 60\left(\left(\frac{4}{9}\right)^1\right) = 10 + \frac{80}{3}$$

$$= 10 + 26\tfrac{2}{3} = 36\tfrac{2}{3}\,°F$$

$$\boxed{T(1) = 36\tfrac{2}{3}\,°F}$$

2) $T(t) = 15°F$, $t = ?$

$$15 = 10 + 60e^{\ln\left(\frac{4}{9}\right)t}$$

$$\frac{5}{60} = e^{\ln\left(\frac{4}{9}\right)t}$$

$$\frac{\ln\left(\frac{1}{12}\right)}{\ln\left(\frac{4}{9}\right)} = t$$

$$\boxed{T(3.06\text{ min}) = 15°F}$$

2)

$$\frac{dA}{dt} = \underbrace{\text{input rate of salt}}_{R_{in}} - \underbrace{\text{output rate of salt}}_{R_{out}}$$

$A(t)$ = amt of salt in container at time t.

$A(0) = 30$

Rate of flow $= 4 \frac{L}{min}$

200 liters of water in tank.

$$R_{in} = 1 \frac{g}{L} \cdot 4 \frac{L}{min} = 4 \frac{g}{min}$$

$$R_{out} = \frac{A}{200} \frac{g}{L} \cdot 4 \frac{L}{min} = \frac{A}{50} \frac{g}{min}$$

$$\boxed{\frac{dA}{dt} = 4 - \frac{A}{50}}$$

$$\frac{dA}{dt} + \frac{1}{50} A = 4$$

Linear Eqn

$$\mu(t) = e^{\int \frac{1}{50} dt} = e^{\frac{1}{50}t}$$

$$e^{\frac{1}{50}t}\left(\frac{dA}{dt} + \frac{1}{50} A\right) = 4e^{\frac{1}{50}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{50}t} A\right) = 4e^{\frac{1}{50}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{50}t} A\right) dt = \int 4e^{\frac{1}{50}t} dt$$

$$e^{\frac{1}{50}t} A = 4\left(e^{\frac{1}{50}t} \cdot 50 + C\right)$$

$$\boxed{A(t) = 200 + C e^{-\frac{1}{50}t}}$$

$A(0) = 200 + C = 30 \rightarrow \boxed{C = -170}$

$$\boxed{A(t) = 200 - 170 e^{-\frac{1}{50}t}}$$

3) <u>Second-order Chemical Reaction</u>

$$\left.\begin{array}{l} A(t) \equiv \text{amt of chemical } A \text{ at time } t \\ B(t) \equiv \text{amt of chemical } B \text{ at time } t \\ C(t) \equiv \text{amt of chemical } C \text{ at time } t \end{array}\right\} \text{in grams.}$$

2 grams of A + 1 gram of B $\rightarrow$ 3 grams of C.

i.e. 2 to 1 ratio grams of A to gram of B in reaction.

$A(0) = 100$

$B(0) = 50$

$C(0) = 0$

$C(10) = 10$

$C(t) = ?$

$C(\infty) \rightarrow ?$

$C(t) = \frac{1}{2}$ amt of $C(\infty)$

$C(t) = 75$, $t = ?$

$$\frac{dC}{dt} \propto A(t)\,B(t)$$

$$\frac{dC}{dt} = k\,A(t)\,B(t)$$

Write $A(t)$ and $B(t)$ in terms of $C(t)$.

Let's assume c grams of chemical C have been made from a grams of A and b grams of B. i.e $c = a + b$. We know that there is a 2-to-1 ratio for a-to-b. So $2a = b$. Then $c = a + 2a$ and $a = \frac{c}{3}$ have been used to make c grams of chemical C. Similarly, $c = \frac{1}{2}b + b$ and $b = \frac{2}{3}c$ have been used to make c grams of chemical C.

Then $\boxed{\frac{dC}{dt} = k\left(100 - \frac{1}{3}C\right)\left(50 - \frac{2}{3}C\right)}$ Separable

$$\frac{dC}{dt} = k(300 - C)(75 - C)$$ Multiply by 3 and $\frac{3}{2}$ (constants are absorbed into k)

$$\frac{1}{(300-C)(75-C)}\frac{dC}{dt} = k$$

partial fractions

$$\frac{1}{(300-c)(75-c)} = \frac{A}{300-c} + \frac{B}{75-c}$$

$$1 = A(75-c) + B(300-c)$$

$$c = 75 \rightarrow \quad 1 = B(300-75)$$

$$\frac{1}{225} = B$$

$$c = 300 \rightarrow \quad 1 = A(75-300)$$

$$\frac{1}{-225} = A$$

$$\int \frac{-\frac{1}{225}}{300-c} + \frac{\frac{1}{225}}{75-c}\, dt$$

$$-\frac{1}{225}\frac{\ln|300-c|}{-1} + \frac{1}{225}\frac{\ln|75-c|}{-1}$$

$$\frac{1}{225}\ln\left(\frac{|300-c|}{|75-c|}\right)$$

Then we have that

$$\frac{d}{dt}\left(\frac{1}{225}\ln\left(\frac{|300-c|}{|75-c|}\right)\right) = k$$

Integrate both sides wrt t.

$$\frac{1}{225}\ln\left(\frac{|300-c|}{|75-c|}\right) = kt + C$$

$$\ln\left(\frac{|300-c|}{|75-c|}\right) = 225kt + C$$

$$\frac{|300-C|}{|75-C|} = e^{225kt+C}$$

absolute values get absorbed into the constant.

$$\frac{300-C}{75-C} = C_0 e^{225kt}$$

$$300-C = C_0 e^{225kt}(75-C)$$

$$-C + C C_0 e^{225kt} = -300 + C_0 e^{225kt} 75$$

$$C(C_0 e^{225kt}-1) = 75(-4 + C_0 e^{225kt})$$

$$\boxed{C(t) = \frac{75(-4 + C_0 e^{225kt})}{C_0 e^{225kt}-1}}$$

$\boxed{C(0)=0}$

$$C(0) = \frac{75(-4+C_0)}{C_0-1} = 0$$

$$\boxed{4 = C_0}$$

$$C(t) = \frac{75(-4+4e^{225kt})}{4e^{225kt}-1}$$

$C(10)=10$

$$C(10)=\frac{75\left(-4+4e^{2250k}\right)}{4e^{2250k}-1}=10$$

$$300\left(-1+e^{2250k}\right)=10\left(4e^{2250k}-1\right)$$

$$e^{2250k}(300-40)=(+300-10)$$

$$e^{2250k}=\frac{290}{260}$$

$$k=\frac{\ln\left(\frac{290}{260}\right)}{2250}$$

$k=4.853E-5$ $\quad$ $k225=1.091E-2$

$$\boxed{C(t)=\frac{300\left(-1+e^{(1.091E-2)t}\right)}{4e^{(1.091E-2)t}-1}}$$

multiply by

$$\frac{e^{(-1.091E-2)t}}{e^{(-1.091E-2)t}}$$

$\downarrow$

$$\boxed{C(t)=\frac{300\left(-e^{(-1.091E-2)t}+1\right)}{4-e^{(-1.091E-2)t}}}$$

$$C(\infty) \longrightarrow \frac{300(1)}{4} = 75$$

$$C(t) = \frac{1}{2} 75 = 37.5 \quad , \quad t = ?$$

$$37.5 = \frac{300\left(-e^{(-1.091E-2)t} + 1\right)}{4 - e^{(-1.091E-2)t}}$$

$$e^{(-1.091E-2)t}(-37.5 + 300) = 300 - 4(37.5)$$

$$(-1.091E-2)\, t = \ln\left(\frac{300 - 4(37.5)}{-37.5 + 300}\right)$$

$$t = 29.3 \text{ min}$$

$$\boxed{C(29.3) \sim 37.5}$$