Math 527
UNH Spring 2015
HW 3 solutions.

1.) $\underbrace{2x-1}_{M(x,y)} + \underbrace{(3y+7)}_{N(x,y)} \frac{dy}{dx} = 0.$

Test for exactness : $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$

$$0 = 0 \checkmark$$

The eqn. is exact ; to solve it, we'll use separation of variables :

$$(3y+7)\frac{dy}{dx} = 1-2x$$

$$\int (3y+7)\, dy = \int (1-2x)\, dx$$

$$\boxed{\frac{3}{2}y^2 + 7y = x - x^2 + C}$$

We'll leave the soln. in implicit form.

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2.) $2x + y - (x + 6y)\frac{dy}{dx} = 0$

$$\underbrace{2x + y}_{M(x,y)} + \underbrace{(-x - 6y)}_{N(x,y)}\frac{dy}{dx} = 0$$

Test for exactness : $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$

$$1 \neq -1 .$$

$\boxed{\text{The eqn. is not exact.}}$ //

3.) $\underbrace{5x + 4y}_{M(x,y)} + \underbrace{(4x - 8y^3)}_{N(x,y)} \frac{dy}{dx} = 0$

Test for exactness : $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$.

$4 = 4$ ✓

The eqn. is exact.

$\therefore \exists\ f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$.

$$\frac{\partial f}{\partial x} = M \Rightarrow \frac{\partial f}{\partial x} = 5x + 4y$$

$$\int \frac{\partial f}{\partial x}\,dx = \int (5x + 4y)\,dx$$

$$f(x,y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{4x} + g'(y) = \cancel{4x} - 8y^3$$

$$\int g'(y)\,dy = \int -8y^3\,dy$$

$$g(y) = -2y^4$$

$\therefore f(x,y) = \frac{5}{2}x^2 + 4xy - 2y^4$

Soln. to the ODE :

$$\boxed{\frac{5}{2}x^2 + 4xy - 2y^4 = C}$$

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4.) $\underbrace{\sin y - y\sin x}_{M(x,y)} + \underbrace{(\cos x + x\cos y - y)}_{N(x,y)}\frac{dy}{dx} = 0.$

Test for exactness : $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$

$$\cos y - \sin x = -\sin x + \cos y \quad ✓$$

The eqn. is exact.

$\therefore \exists\ f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N.$

$$\frac{\partial f}{\partial x} = M \Rightarrow \frac{\partial f}{\partial x} = \sin y - y\sin x$$

$$\int \frac{\partial f}{\partial x}\,dx = \int (\sin y - y\sin x)\,dx$$

$$f(x,y) = x\sin y + y\cos x + g(y)$$

$$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{x\cos y} + \cancel{\cos x} + g'(y) = \cancel{\cos x} + \cancel{x\cos y} - y$$

$$\int g'(y)\,dy = \int -y\,dy$$

$$g(y) = -\frac{y^2}{2}$$

$\therefore\ f(x,y) = x\sin y + y\cos x - \frac{y^2}{2}$

Soln. to ODE :

$$\boxed{x\sin y + y\cos x - \frac{y^2}{2} = C}$$

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5.) $\underbrace{x^2 - y^2}_{M(x,y)} + \underbrace{(x^2 - 2xy)}_{N(x,y)} \frac{dy}{dx} = 0$

Test for exactness : $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$

$$-2y \neq 2x - 2y$$

$\boxed{\text{The eqn. is not exact.}}$

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6.) $\underbrace{x^2 y^3 - \frac{1}{1+9x^2}}_{M(x,y)} + \underbrace{x^3 y^2}_{N(x,y)} \frac{dy}{dx} = 0$

Test for exactness: $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$

$$3x^2y^2 = 3x^2y^2 \quad \checkmark$$

The eqn. is exact.

$\therefore \exists \ f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$.

$$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial f}{\partial y} = x^3 y^2$$

$$\int \frac{\partial f}{\partial y}\, dy = \int x^3 y^2 \, dy$$

$$f(x,y) = \frac{1}{3} x^3 y^3 + g(x)$$

$$\frac{\partial f}{\partial x} = M \Rightarrow \cancel{x^2 y^3} + g'(x) = \cancel{x^2 y^3} - \frac{1}{1+9x^2}$$

$$\int g'(x)\, dx = -\int \frac{1}{1+9x^2}\, dx$$

$$= -\int \frac{1}{1+(3x)^2}\, dx$$

$$g(x) = -\frac{1}{3}\tan^{-1}(3x)$$

$\therefore \ f(x,y) = \frac{1}{3}x^3y^3 - \frac{1}{3}\tan^{-1}(3x)$

Soln. to ODE:

$$\boxed{\frac{1}{3}x^3y^3 - \frac{1}{3}\cdot\tan^{-1}(3x) = C}$$

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7.) $2y\sin x\cos x - y + 2y^2 e^{xy^2} = \left(x - \sin^2 x - 4xye^{xy^2}\right)\frac{dy}{dx}$

$$\underbrace{2y\sin x\cos x - y + 2y^2e^{xy^2}}_{M(x,y)} + \underbrace{\left(\sin^2 x - x + 4xye^{xy^2}\right)}_{N(x,y)}\frac{dy}{dx} = 0$$

Test for exactness : $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$

Yes ; both sides equal $2\sin x\cos x - 1 + 4ye^{xy^2} + 4xy^3e^{xy^2}$.

$\therefore\ \exists\ f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$.

$\frac{\partial f}{\partial x} = M \Rightarrow \frac{\partial f}{\partial x} = 2y\sin x\cos x - y + 2y^2e^{xy^2}$

$$\int \frac{\partial f}{\partial x}\,dx = \int\left(2y\sin x\cos x - y + 2y^2e^{xy^2}\right)dx$$

$$f(x,y) = y\sin^2 x - xy + 2e^{xy^2} + g(y)$$

$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{\sin^2 x} - \cancel{x} + \cancel{4xye^{xy^2}} + g'(y) = \cancel{\sin^2 x} - \cancel{x} + \cancel{4xye^{xy^2}}$

$$\int g'(y)\,dy = \int 0\,dy$$

$$g(y) = 0$$

$\therefore\ f(x,y) = y\sin^2 x - xy + 2e^{xy^2}$

Soln. to ODE:

$$\boxed{y\sin^2 x - xy + 2e^{xy^2} = C}$$

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8.) $t\,\frac{dy}{dt} = 2te^{t} - y + 6t^2$

$$\underbrace{y - 2te^{t} - 6t^2}_{M(t,y)} + \underbrace{t}_{N(t,y)}\frac{dy}{dt} = 0$$

Test for exactness : $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial t}$

$$1 = 1 \quad \checkmark$$

The eqn. is exact.

$\therefore\ \exists\ f(t,y)$ s.t. $\frac{\partial f}{\partial t} = M$ & $\frac{\partial f}{\partial y} = N.$

$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial f}{\partial y} = t$

$$\int \frac{\partial f}{\partial y}\,dy = \int t\,dy$$

$$f(t,y) = ty + g(t)$$

$\frac{\partial f}{\partial t} = M \Rightarrow \cancel{y} + g'(t) = \cancel{y} - 2te^{t} - 6t^2$

$$\int g'(t)\,dt = -2\underbrace{\int te^{t}\,dt}_{\text{integration by parts}} - 6\int t^2\,dt$$

$$g(t) = -2e^{t}(t-1) - 2t^3$$

$\therefore\ f(t,y) = ty - 2e^{t}(t-1) - 2t^3$

Soln. to ODE:

$$\boxed{ty - 2e^{t}(t-1) - 2t^3 = C}$$

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9.) $(x+y)^2 + (2xy + x^2 - 1)\frac{dy}{dx} = 0$, $\quad y(1) = 1$

$$\underbrace{x^2 + 2xy + y^2}_{M(x,y)} + \underbrace{(2xy + x^2 - 1)}_{N(x,y)}\frac{dy}{dx} = 0$$

Test for exactness : $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$

$$2x + 2y = 2y + 2x \quad ✓$$

The eqn. is exact.
$\therefore \ \exists \ f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$.

$\frac{\partial f}{\partial x} = M \Rightarrow \frac{\partial f}{\partial x} = x^2 + 2xy + y^2$

$$\int \frac{\partial f}{\partial x}\,dx = \int (x^2 + 2xy + y^2)\,dx$$

$$f(x,y) = \frac{1}{3}x^3 + x^2y + xy^2 + g(y)$$

$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{x^2} + \cancel{2xy} + g'(y) = \cancel{2xy} + \cancel{x^2} - 1$

$$\int g'(y)\,dy = \int (-1)\,dy$$

$$g(y) = -y$$

$\therefore \ f(x,y) = \frac{1}{3}x^3 + x^2y + xy^2 - y$.
General soln. to ODE:

$$\frac{1}{3}x^3 + x^2y + xy^2 - y = C$$

Use $y(1) = 1$ to solve for C: $\frac{1}{3} + 1 = C$

$$\frac{4}{3} = C$$

Soln. to IVP: $\boxed{\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}}$ //

10.) $\underbrace{e^x + y}_{M(x,y)} + \underbrace{(2 + x + ye^y)}_{N(x,y)} \frac{dy}{dx} = 0$, $y(0) = 1$

Test for exactness : $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$.

$1 = 1$ ✓

The eqn. is exact.
$\therefore \exists\ f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$.

$\frac{\partial f}{\partial x} = M \Rightarrow \frac{\partial f}{\partial x} = e^x + y$

$$\int \frac{\partial f}{\partial x}\, dx = \int (e^x + y)\, dx$$

$$f(x,y) = e^x + xy + g(y)$$

$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{x} + g'(y) = 2 + \cancel{x} + ye^y$

$$\int g'(y)\, dy = \int 2\, dy + \int ye^y\, dy$$

$$g(y) = 2y + e^y(y-1)$$

$\therefore f(x,y) = e^x + xy + 2y + e^y(y-1)$

General soln. to ODE: $e^x + xy + 2y + e^y(y-1) = C$

Use $y(0) = 1$ to solve for C: $1 + 2 = C \Rightarrow 3 = C$

Soln. to IVP : $\boxed{e^x + xy + 2y + e^y(y-1) = 3}$ //.