Solutions of HW2 MATH 527

1. Solution. $\frac{dy}{dx} + y = e^{3x}$ This is already the standard form of 1^{st} ODE.

$p(x) = 1$. Integrating factor $u(x) = e^{\int p(x)dx} = e^{x}$ [pick the easiest form of $u(x)$]

① $u(x)\left[\frac{dy}{dx} + y\right] = u(x)e^{3x} \Rightarrow \frac{d}{dx}[u(x)y] = u(x)e^{3x}$

② $u(x)y = \int e^{x}e^{3x}dx + C = \int e^{4x}dx + C$

③ $y = \frac{1}{u(x)}\left[\int e^{4x}dx + c\right] = e^{-x}\left[\frac{1}{4}e^{4x} + C\right] = \frac{1}{4}e^{3x} + Ce^{-x}$

$-\infty < x < \infty$

2. Solution. $\frac{dy}{dx} + 2xy = x^3$. $p(x) = 2x$. $u(x) = e^{\int p(x)dx} = e^{x^2}$

① $u(x)\left[\frac{dy}{dx} + 2xy\right] = u(x)\cdot x^3 \Rightarrow \frac{d}{dx}[u(x)y] = u(x)\cdot x^3$

② $u(x)y = \int e^{x^2}\cdot x^3 dx + C_1$

$$\int e^{x^2}\cdot x^3 dx = \frac{1}{2}\int e^{x^2}\cdot x^2 dx^2 \overset{x^2=v}{=\!=} \frac{1}{2}\int e^{v}\cdot v\,dv$$

Integration by parts: $\frac{1}{2}\int e^{v}\cdot v\,dv = \frac{1}{2}\int v\,de^{v} = \frac{1}{2}\left[ve^{v} - \int e^{v}dv\right]$

$= \frac{1}{2}\left[ve^{v} - e^{v} + C_2\right]$, plug in $v = x^2$

③ $\therefore u(x)y = \frac{1}{2}\left[x^2e^{x^2} - e^{x^2} + C_2\right] + C_1 = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + \frac{1}{2}C_2 + C_1$

set $C = \frac{1}{2}C_2 + C_1$, $y = \frac{1}{2}x^2 - \frac{1}{2} + C\cdot e^{-x^2}$, $-\infty < x < \infty$

3. Solution. $x\frac{dy}{dx} - y = x^2\sin x$, if $x \neq 0 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = x\sin x$

$\therefore p(x) = -\frac{1}{x}$, $u(x) = e^{\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

① $u(x)\left[\frac{dy}{dx} - \frac{1}{x}y\right] = u(x)\,x\sin x \Rightarrow \frac{d}{dx}[u(x)y] = u(x)\,x\sin x$

② $u(x)y = \int u(x)\,x\sin x\,dx + C_1 = \int \sin x\,dx + C_1 = -\cos x + C_2 + C_1$,

set $C = C_2 + C_1$, ③ $y = x\cdot[-\cos x + C]$ $(*)$

check $x = 0$, L.H.S. $= -y$, R.H.S. $= 0$ $\therefore y = 0$, which is included in $(*)$

$\therefore x \in (-\infty, \infty)$

4. Solution. if $\cos x \neq 0$, $x \neq \frac{\pi}{2} + k\pi$, $k = 0, \pm1, \pm2, \ldots \Rightarrow \frac{dy}{dx} + \frac{\sin x}{\cos x}y = \frac{1}{\cos x}$

$\therefore p(x) = \frac{\sin x}{\cos x}$, $u(x) = e^{\int p(x)dx} = e^{\int \frac{\sin x}{\cos x}dx} = \frac{1}{\cos x}$

① $u(x)\left[\frac{dy}{dx} + \frac{\sin x}{\cos x}y\right] = u(x)\frac{1}{\cos x} \Rightarrow \frac{d}{dx}[u(x)y] = u(x)\frac{1}{\cos x}$

② $u(x)y = \int u(x)\frac{1}{\cos x}dx + C_1$, $\int u(x)\frac{1}{\cos x}dx = \int \frac{1}{\cos^2 x}dx = \int \sec^2 x\,dx = \tan x + C_2$

$\therefore u(x)y = \tan x + C$, $C = C_2 + C_1$, $y = \sin x + C\cos x$

Check $\cos x = 0$, $x = \frac{\pi}{2} + k\pi$, $k = 0, \pm1, \pm2, \ldots$ $y = \sin x$, $\sin^2 x = 1$ ✓. $\therefore x \in (-\infty, \infty)$

5. Solution. $\frac{dy}{dx} + 3y = 2x$, $y(0) = \frac{1}{3}$, $p(x) = 3$. $u(x) = e^{\int p(x)dx} = e^{3x}$

① $u(x)\left[\frac{dy}{dx} + 3y\right] = u(x)\cdot 2x \Rightarrow \frac{d}{dx}[u(x)y] = u(x)\cdot 2x$

② $u(x)y = \int u(x)\cdot 2x\,dx + C_1$, $\int u(x)\cdot 2x\,dx = \int e^{3x}\cdot 2x\,dx$ Integration by parts

$\int e^{3x}\,2x\,dx = \frac{2}{3}\int x\,de^{3x} = \frac{2}{3}\left[xe^{3x} - \int e^{3x}dx\right] = \frac{2}{3}\left[xe^{3x} - \frac{1}{3}e^{3x} + C_2\right]$

③ $y = e^{-3x}\left[\frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + \frac{2}{3}C_2 + C_1\right]$, set $\frac{2}{3}C_2 + C_1 = C$

$= \frac{2}{3}x - \frac{2}{9} + Ce^{-3x}$, $y(0) = -\frac{2}{9} + C\cdot 1 = \frac{1}{3} \Rightarrow C = \frac{5}{9}$

$\therefore\ y = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}$, $-\infty < x < +\infty$

6. Solution. $t\frac{dy}{dt} + y = e^t$, $y(1) = 2$. if $t \neq 0$, $\Rightarrow \frac{dy}{dt} + \frac{1}{t}y = \frac{e^t}{t}$, $p(t) = \frac{1}{t}$

$u(t) = e^{\int p(t)dt} = e^{\int \frac{1}{t}dt} = t$, ① $u(t)\left[\frac{dy}{dt} + \frac{1}{t}y\right] = \frac{d}{dt}[u(t)y] = u(t)\frac{e^t}{t}$

② $u(t)y = \int e^t dt = e^t + C$ ③ $y = \frac{e^t}{t} + \frac{C}{t}$, $y(1) = \frac{e}{1} + C = 2$, $\therefore C = 2 - e$

$y(t) = \frac{e^t}{t} + \frac{2-e}{t}$ check $t = 0$, $y = 1$; $t \neq 0$, $y(t) = \frac{e^t}{t} + \frac{2-e}{t}$, $t=0$ is a singular point.

7. Solution. $(x+1)\frac{dy}{dx} + y = \ln x$, $\therefore x > 0$, $\Rightarrow \frac{dy}{dx} + \frac{1}{x+1}y = \frac{1}{x+1}\ln x$, $u(x) = e^{\int \frac{1}{x+1}dx} = x+1$

① $u(x)\left[\frac{dy}{dx} + \frac{1}{x+1}y\right] = \frac{d}{dx}[u(x)y] = u(x)\cdot\frac{1}{x+1}\ln x = \ln x$

② $u(x)y = \int \ln x\,dx = x\ln x - x + C$ ③ $y = \frac{1}{x+1}[x\ln x - x + C]$

$y(1) = \frac{1}{2}[C-1] = 10$ $\therefore C = 21$, $y = \frac{1}{x+1}[x\ln x - x + 21]$, $x \in (0, +\infty)$

8. Solution. $x(x+1)\frac{dy}{dx} + xy = 1$, if $x(x+1) \neq 0$, $x \neq 0$, $x \neq -1$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x+1}y = \frac{1}{x(x+1)}$, $p(x) = \frac{1}{x+1}$, $u(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x+1}dx} = x+1$

① $u(x)\cdot\left[\frac{dy}{dx} + \frac{1}{x+1}y\right] = \frac{d}{dx}[u(x)y] = \frac{1}{x}$

② $u(x)\cdot y = \int \frac{1}{x}dx = \ln|x| + C$

③ $y = \frac{1}{x+1}[\ln|x| + C]$, $y(e) = \frac{1+C}{e+1} = 1$ $\therefore C = e$

$y = \frac{1}{x+1}[\ln|x| + e]$, $x \neq 0$, $x \neq -1$.

check $x = 0$, L.H.S $= 0$, R.H.S $= 1$, $\therefore x \neq 0$, or there is no solution.

check $x = -1$, L.H.S $= -y$, R.H.S $= 1 \Rightarrow y(-1) = -1$, It's a singular points.