Math 527
SPRING 2015
HW#1 SOLNS

Disclaimer:

The method I used in separation of variables is not what Professor Gibson uses in class. However, the idea and result are the same.

i.e., Problem 1.

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x}$$

Method from class

$$\int \frac{dy}{dx}\frac{1}{y}\,dx = \int \frac{4}{x}\,dx$$

$$\int \frac{d}{dx}\left(\ln|y|\right)dx = \int \frac{4}{x}\,dx.$$

$$\ln|y| = \int \frac{4}{x}\,dx.$$

My method

$$y = \phi(x)$$

$$\frac{dy}{dx} = \phi'(x)$$

$$\frac{1}{\phi(x)}\phi'(x) = \frac{4}{x}$$

$$\int \frac{1}{\phi(x)}\underbrace{\phi'(x)\,dx}_{dy} = \int \frac{4}{x}\,dx.$$

$$\int \frac{1}{y}\,dy = \int \frac{4}{x}\,dx$$

$$\ln|y| = \int \frac{4}{x}\,dx.$$

Separation of Variables

1) $x\dfrac{dy}{dx} = 4y$

Put the problem in proper form.

$$\frac{dy}{dx} = \frac{4}{x}\cdot y$$

Separate the variables.

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x} \qquad (y \neq 0)$$

(Intermediate steps)

Let $y = \phi(x)$.

$$\frac{dy}{dx} = \phi'(x)$$

$$\frac{1}{\phi(x)}\phi'(x) = \frac{4}{x}$$

Integrate both sides wrt x.

$$\int \frac{1}{\phi(x)}\phi'(x)\,dx = \int \frac{4}{x}\,dx$$

Rewrite LHS in terms of y.

$$\int \frac{1}{y}\,dy = \int \frac{4}{x}\,dx.$$

$$\ln|y| = 4\ln|x| + C_0$$

Exponentiate.

$$|y| = e^{4\ln|x| + C_0}$$

$$|y| = e^{\ln(x^4)}e^{C_0}$$

$$\boxed{y = 0 \rightarrow C_1 = 0}$$

$$y = \pm e^{C_0}(x^4) \rightarrow \boxed{y(x) = C_1 x^4}$$

2) $\dfrac{dy}{dx} = e^{3x+2y}$

proper form.

$$\frac{dy}{dx} = e^{3x}e^{2y}$$

Separate the variables.

$$e^{-2y}\frac{dy}{dx} = e^{3x}$$

(Int. steps)

Let $y = \Phi(x)$ $\quad \dfrac{dy}{dx} = \Phi'(x)$

$$e^{-2\Phi(x)}\Phi'(x) = e^{3x}$$

Integrate both sides wrt x.

$$\int e^{-2\phi(x)}\phi'(x)\,dx = \int e^{3x}\,dx.$$

Rewrite LHS in terms of y.

same as in class.

$$\int \frac{dy}{dx}e^{-2y}\,dx$$

$$\int \frac{d}{dx}\left(-\tfrac{1}{2}e^{-2y}\right)dx.$$

$$\int e^{-2y}\,dy = \int e^{3x}\,dx.$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C_0$$

$$e^{-2y} = -\frac{2}{3}e^{3x} + C_1 \qquad (C_1 = -2C_0)$$

$$\ln\left(e^{-2y}\right) = \ln\left(-\frac{2}{3}e^{3x} + C_1\right)$$

$$-2y = \ln\left|-\frac{2}{3}e^{3x} + C_1\right|$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left|-\frac{2}{3}e^{3x} + C_1\right|}$$

3) $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$

proper form

$$\frac{dy}{dx} = \frac{1}{(4x+5)^2} \cdot (2y+3)^2$$

Separate the variables.

$$\frac{1}{(2y+3)^2}\frac{dy}{dx} = \frac{1}{(4x+5)^2} \qquad y \neq -\tfrac{3}{2}$$

Intermediate Steps.

Let $y = \phi(x)$ $\dfrac{dy}{dx} = \phi'(x)$.

$$(2\phi(x)+3)^{-2}\,\phi'(x) = (4x+5)^{-2}$$

Int wrt x.

$$\int (2\phi(x)+3)^{-2}\,\phi'(x)\,dx = \int (4x+5)^{-2}\,dx.$$

Rewrite LHS in terms of y.

$$\int (2y+3)^{-2}\,dy = \int (4x+5)^{-2}\,dx.$$

$$-\frac{(2y+3)^{-1}}{2} = -\frac{(4x+5)^{-1}}{4} + C_0$$

Chain rule

Solve for $y(x)$. (combine fractions)

$$-\frac{1}{2(2y+3)} = -\frac{(1+C_1)}{4(4x+5)} \qquad C_1 = -C_0 4(4x+5)$$

Cross multiply.

$$4(4x+5) = 2(2y+3)(1+C_1)$$

$$16x+20 = (4y+6)(1+C_1)$$

To solve for y, divide by $(1+C_1)$

$$16x+20 = (4y+6)(1+C_1)$$

$$\frac{16x+20}{1+C_1} = 4y+6$$

$y = -\frac{3}{2} \rightarrow$ no soln for C_1

$$\boxed{\frac{1}{4}\left[\frac{16x+20}{1+C_1} - 6\right] = y(x)}$$

for $y \neq -\frac{3}{2}$.

Notes for problem 3

I. $(2y+3)^2 = 4y^2+12y+9$

$$\int \frac{1}{4y^2+12y+9}\,dy$$

$$\neq \frac{\ln\left|4y^2+12y+9\right|}{(8y+12)} + C.$$

But it is true that...

$$\int \frac{1}{y+a}\,dy = \ln|y+a| + c.$$

(y must have a power of 1 in the denominator to use this formula).

Notes, problem 3.

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HW # 1 Solns.

II. $\frac{dy}{dx} = \frac{4y^2+12y+9}{16x^2+40x+25}$

$\vdots$

$$\int \frac{1}{4y^2+12y+9}\,dy = \int \frac{1}{16x^2+40x+25}\,dx$$

$\boxed{\text{Partial Fractions DOES NOT HELP.}}$

LHS

$$\frac{1}{(2y+3)(2y+3)} = \frac{A}{2y+3} + \frac{B}{(2y+3)^2}$$

Make common denominator on both sides.

$$1 = A(2y+3) + B$$

If $y = -\frac{3}{2}$ then $2\left(-\frac{3}{2}\right)+3 = 0$

$$1 = B.$$

then $1 = A(2y+3) + 1$

$$A = 0.$$

$$\int \frac{1}{4y^2+12y+9}\,dy = \int \frac{1}{(2y+3)^2}\,dy.$$

4) $\csc(y) + \sec^2(x)\frac{dy}{dx} = 0$

get to proper form.

$$\frac{dy}{dx} = -\csc(y)\cdot\frac{1}{\sec^2(x)}$$

separate the variables.

$$-\frac{1}{\csc(y)}\frac{dy}{dx} = \frac{1}{\sec^2(x)}$$

skip intermediate steps.
(see problems #1-3)

$$-\int\frac{1}{\csc(y)}\,dy = \int\frac{1}{\sec^2(x)}\,dx$$

$$-\int\sin(y)\,dy = \int\cos^2(x)\,dx$$

LHS: $\cos(y) + C_0$

RHS: Approach #1 IBP (Reduction Formula)

$u = \cos(x)$ $\quad v = +\sin(x)$

$du = -\sin(x)\,dx$ $\quad dv = \cos(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int\cos^2(x)\,dx = \cos(x)\sin(x) + \int\sin^2(x)\,dx.$$

$$\quad\quad = \quad\quad + \int(1-\cos^2(x))\,dx.$$

Let $\int\cos^2(x)\,dx = I$. Then

$$I = \cos(x)\sin(x) + \int 1\,dx - I.$$

$$2I = \cos(x)\sin(x) + x + C_0$$

$$2I = \cos(x)\sin(x) + x + C_0$$

$$\int\cos^2(x)\,dx = \boxed{\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C_1}$$

RHS: Approach #2 trig identity

$$\cos^2(x) = \frac{1}{2}(1+\cos 2x)$$

$$\int\cos^2(x)\,dx \equiv \int\left(\frac{1}{2}(1+\cos 2x)\right)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int\cos 2x\,dx.$$

$$= \frac{1}{2}x + \frac{1}{2}\sin(2x)\frac{1}{2} + C_1$$

$$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C_1}$$

Note that

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

App. #1 = App. #2

$$\frac{1}{2}\cos(x)\sin(x) = \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)$$

$$\therefore\ \cos(y) = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C_2$$

$$\boxed{y(x) = \cos^{-1}\left(\cos(x)\sin(x) + x + C_3\right)}$$

$$C_3 = 2C_2$$

5) $\frac{dP}{dt} = P - P^2$

proper form $\frac{dP}{dt} = g(t)h(P)$

$$\frac{dP}{dt} = 1 \cdot (P - P^2)$$

Sep. variables.

$$\frac{1}{P-P^2}\frac{dP}{dt} = 1$$

Chain Rule!

$$\vdots$$

$$\int \frac{1}{P-P^2}\,dP = \int 1\,dt$$

$P(1-P)$ #1 partial fractions

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$1 = A(1-P) + BP$

$P=1 \rightarrow B=1$

$1 = A(1-P) + P$

$P=0 \rightarrow A=1$

$$\int \frac{1}{P}\,dP + \int \frac{1}{1-P}\,dP = \int 1\,dt$$

#2

$$\frac{1}{P(1-P)} = \frac{1-P+P}{P(1-P)}$$

$$= \frac{(1-P)}{P(1-P)} + \frac{P}{P(1-P)}$$

$$= \frac{1}{P} + \frac{1}{1-P}.$$ same result!

$$\int \frac{1}{P}\,dP + \int \frac{1}{1-P}\,dP = \int 1\,dt$$

$$\ln|P| - \ln|1-P| = t + C_0$$

$$\ln\left(\frac{|P|}{|1-P|}\right) = t + C_0$$

$$\frac{|P|}{|1-P|} = e^t e^{C_0}$$

(Technically, here you can drop the absolute value into C_1 but here is the reasoning if you didn't.)

$$\frac{|1-P|}{|P|} = C_1 e^{-t}$$

Consider $|1-P| < \begin{matrix} 1-P \\ P-1 \end{matrix}$

$0 < P < 1 \rightarrow |1-P| \equiv 1-P$, $|P| \equiv P$

$$\frac{1-P}{P} = C_1 e^{-t}$$

$$1 = C_1 P e^{-t} + P$$

$$\boxed{\frac{1}{C_1 e^{-t} + 1} = P(t)}$$

$P > 1 \rightarrow |1-P| \equiv P-1$, $|P| \equiv P$

$$\frac{P-1}{P} = C_1 e^{-t}$$

$$P = P C_1 e^{-t} + 1$$

$$P(1 - C_1 e^{-t}) = 1$$

$$\boxed{P(t) = \frac{1}{1 - C_1 e^{-t}}}$$

$-1 < P < 0 \rightarrow |P| \equiv -|P|$

$|1-P| \equiv 1-(-|P|)$

$1+|P|$

$$\frac{1+|P|}{-|P|} = C_1 e^{-t}$$

$$1 = -|P| - |P| C_1 e^{-t}$$

$$1 = -|P|\left(1 + C_1 e^{-t}\right)$$

$$-|P| = \frac{1}{1+C_1 e^{-t}}$$

$$\boxed{P(t) = \frac{1}{1+C_1 e^{-t}}}$$

$-1 < P \rightarrow |P| \equiv -|P|$

$|1-P| \equiv 1-(-|P|)$

$\boxed{\text{same as previous case}}$

Conclusion

$$\boxed{\begin{array}{l} \text{for } t>0, \\ P(t) = \dfrac{1}{1+Ce^{-t}} \\ \text{Note, same as} \\ P(t) = \dfrac{Ce^{t}}{Ce^{t}+1} \end{array}}$$

(this is "logistic growth")

6) IVP

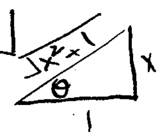
$$\frac{dx}{dt} = 4(x^2+1), \quad x\left(\frac{\pi}{4}\right) = 1$$

$$\int \frac{dx}{x^2+1} = \int 4\,dt$$

trig. sub. $x \neq \pm i$

$$\int \frac{dx}{\left(\sqrt{x^2+1}\right)^2} = \int 4\,dt$$

LHS



$$\frac{x}{1} = \tan\theta$$

$$dx = \sec^2\theta\,d\theta$$

$$\frac{\sqrt{x^2+1}}{1} = \frac{\text{hyp.}}{\text{adj}} = \sec\theta$$

$$\therefore \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \int d\theta = \theta + C_0$$

$$\int \frac{dx}{x^2+1} = \arctan(x) + C_0$$

RHS $\int 4\,dt = 4t + C_1$

Then $\arctan(x) = 4t + C$

$$x(t) = \tan(4t + C)$$

IC: $x\left(\frac{\pi}{4}\right) = 1$

$$x\left(\frac{\pi}{4}\right) = \tan(\pi + C) = 1$$

$C = \arctan(1) - \pi$

$$C = \arctan(1) - \pi$$

What angle θ gives

$\tan\theta = 1$?

$\frac{opp}{adj} = 1$?

$\theta = \frac{\pi}{4}$

$$C = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

Then

$$\boxed{x(t) = \tan\left(4t - \frac{3\pi}{4}\right)}$$

7) $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}, \; y(2) = 2$

$$\int \frac{1}{y^2-1}\,dy = \int \frac{1}{x^2-1}\,dx$$

Can do partial fractions or trig sub. here.

partial fractions

$$\frac{1}{y^2-1} = \frac{A}{(y+1)} + \frac{B}{(y-1)}$$

$$1 = A(y-1) + B(y+1)$$

$y = 1 \rightarrow 1 = B2$

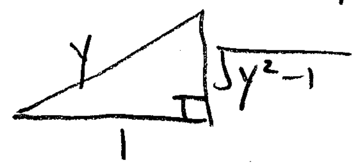
$B = \frac{1}{2}$

$y = -1 \rightarrow 1 = -2A$

$A = -\frac{1}{2}$

trig sub (not the best approach)

$$\int \frac{dy}{y^2-1} = \int \frac{dy}{\sqrt{y^2-1}^{\,2}}$$



$\frac{y}{1} = \sec\theta$

$dy = \sec\theta\tan\theta\, d\theta$

$\frac{\sqrt{y^2-1}}{1} = \tan\theta$

$$\int \frac{\sec\theta\tan\theta\, d\theta}{\tan^2\theta} = \int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta}\frac{1}{\cos\theta}\, d\theta = \int \sin^{-1}\theta\, d\theta$$

$$= \int \frac{\sin\theta}{\sin^2\theta}\, d\theta = \int \frac{\sin\theta}{1-\cos^2\theta}\, d\theta$$

u-sub $u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\int \frac{-du}{1-u^2} \rightsquigarrow$$ still need partial fractions.

$$\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1}$$

$$-\frac{1}{2}\int \frac{dy}{y+1} + \frac{1}{2}\int \frac{dy}{y-1} = -\frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$$

$$-\frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$\frac{1}{2}\ln\left(\frac{|y-1|}{|y+1|}\right) = \frac{1}{2}\ln\left(\frac{|x-1|}{|x+1|}\right) + C$$

$$\frac{1}{2}\ln\left(\frac{|y-1|}{|y+1|}\right) = \frac{1}{2}\ln\left(\frac{|x-1|}{|x+1|}\right) + C_0$$

$$\ln\left(\frac{|y-1|}{|y+1|}\right) = \ln\left(\frac{|x-1|}{|x+1|}\right) + C_1$$

$$\frac{y-1}{y+1} = C_2\,\frac{x-1}{x+1}$$

$y(2) = 2$

$$\frac{2-1}{2+1} = C_2\,\frac{(2-1)}{2+1}$$

$$C_2 = 1$$

$$\boxed{\frac{y-1}{y+1} = \frac{x-1}{x+1}}$$

$\Downarrow$

LHS & RHS fcns are the same.

$$\boxed{y(x) = x.}$$

8) $x^2 \frac{dy}{dx} = y - xy$, $y(-1) = -1$

proper form

$$\frac{dy}{dx} = y\frac{(1-x)}{x^2}$$

Separate variables

$$\frac{1}{y}\frac{dy}{dx} = \frac{1-x}{x^2}$$

$\vdots$

$$\int \frac{1}{y}\,dy = \int \frac{1-x}{x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx - \int \frac{1}{x}\,dx$$

$$\ln|y| = -x^{-1} - \underbrace{\ln|x|}_{-\ln|x| = \ln|x^{-1}|} + C_0$$

$$\cancel{|y| = e^{-\frac{1}{x}} e^{\frac{1}{x}} e^{C_0}}$$

$$|y| = e^{-\frac{1}{x}} e^{\ln\left|\frac{1}{x}\right|} e^{C_0}$$

$$y = \pm e^{C_0} e^{-\frac{1}{x}} \left|\frac{1}{x}\right|$$

$$y = C_1 e^{-\frac{1}{x}} \left|\frac{1}{x}\right|$$

IC $y(-1) = 1$

$$y(-1) = C_1 e^{-\frac{1}{-1}} \left|\frac{1}{-1}\right| = 1$$

$$C_1 e^1 = 1 \rightarrow C_1 = e^{-1}$$

$$\therefore \boxed{y(x) = e^{-1-\frac{1}{x}} \left|\frac{1}{x}\right|} \text{ for } x \text{ finite.}$$

9) $\frac{dy}{dt} + 2y = 1$, $y(0) = \frac{5}{2}$

proper form

$$\frac{dy}{dt} = (1-2y)\cdot(1)$$

Separate variables

$$\frac{1}{1-2y}\frac{dy}{dt} = 1$$

$\vdots$

$$\int \frac{1}{1-2y}\,dy = \int dt \qquad y \neq \tfrac{1}{2}$$

$$\frac{\ln|1-2y|}{-2} = t + C_0$$

$$\ln|1-2y| = -2t + C_1$$

$$|1-2y| = e^{-2t} e^{C_1}$$

$$1-2y = C_2 e^{-2t}$$

$$y(t) = \frac{C_2 e^{-2t} - 1}{-2}$$

$$y(t) = C_3 e^{-2t} + \frac{1}{2}$$

$y(t) = C_3 e^{-2t} + \frac{1}{2}$

$y(0) = C_3 + \frac{1}{2} = \frac{5}{2}$

$C_3 = 2$

$\therefore \boxed{y(t) = 2e^{-2t} + \frac{1}{2}}\ \boxed{\text{for } t \neq 0}$

$y(t) = \frac{1}{2} = 2e^{-2t} + \frac{1}{2}$

$\hookrightarrow t = 0$

$\boxed{y(0) = \frac{1}{2}}$

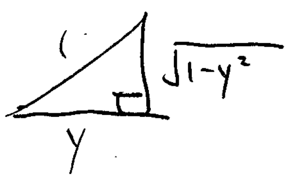
10) $\frac{dy}{dx} = x\sqrt{1-y^2}$

- Already in proper form
- separate variables.

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int x\,dx$$

$y \neq 1, -1$

LHS] trig sub.



$\frac{y}{1} = \cos\theta$

$dy = -\sin\theta\, d\theta$

$\frac{\sqrt{1-y^2}}{1} = \sin\theta$

$$\int \frac{-\sin\theta\, d\theta}{\sin\theta} = \int -d\theta = -\theta + C_0$$

$$= -\arccos(y) + C_0$$

Then

$$-\arccos(y) = \int x\,dx - C_0$$

$$-\arccos(y) = \frac{x^2}{2} + C_1$$

$$\boxed{y(x) = \cos\left(-\frac{x^2}{2} + C_2\right)}$$

Note:

$y = \sin\theta$

$dy = \cos\theta\, d\theta$

$\sqrt{1-y^2} = \cos\theta$

$$\int \frac{\cos\theta\, d\theta}{\cos\theta} = \theta + C_0 = \arcsin(y) + C_0$$

$$\boxed{y(x) = \sin\left(\frac{x^2}{2} + C_1\right)}$$ (same soln. expressed differently).

singular soln

$y = 1 \rightarrow -\frac{x^2}{2} + C_2 = 0$

$$\boxed{x = \pm\sqrt{2C_2}}$$

$y = -1 \rightarrow -\frac{x^2}{2} + C_2 = \pi$

$$\boxed{x \pm \sqrt{2\pi + 2C_2}, \quad C_2 > \pi}$$

$|y| < 1$

$$0 < -\frac{x^2}{2} + C_2 < \pi$$

$$-C_2 < -\frac{x^2}{2} < \pi - C_2$$

$-C_2 < -\frac{x^2}{2}$	$-\frac{x^2}{2} < \pi - C_2$
$C_2 > \frac{x^2}{2}$	$\frac{x^2}{2} > -\pi + C_2$
$\pm\sqrt{2C_2} > x$	$x > \pm\sqrt{2\pi + 2C_2}$, $C_2 > \pi$

$$\boxed{\pm\sqrt{2\pi + 2C_2} < x < \pm\sqrt{2C_2}}$$