HW 11 solns.

1. $x' = -x + 2y$
$y' = -7x + 8y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(8-\lambda) + 14 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$(\lambda - 1)(\lambda - 6) = 0$$

$\lambda_1 = 1$
$\lambda_2 = 6$

Eigenvector for $\lambda_1 = 1$: solve $(A - I)\vec{k}_1 = \vec{0}$

$$\begin{pmatrix} -2 & 2 \\ -7 & 7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2v_1 + 2v_2 = 0$$
$$-7v_1 + 7v_2 = 0$$

$\Rightarrow$ $v_1 = v_2$, only constraint. Let $v_1 = 1 \Rightarrow v_2 = 1$.

So $\vec{k}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 6$: solve $(A - 6I)\vec{k}_2 = \vec{0}$.

$$\begin{pmatrix} -7 & 2 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad -7v_1 + 2v_2 = 0$, only constraint.

Let $v_1 = 2$; then $v_2 = 7$.

So $\vec{k}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

General soln.

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{6t}}$$

(2.)

$$x' = 2x + 2y$$
$$y' = x + 3y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 2 = 0$$
$$6 - 5\lambda + \lambda^2 - 2 = 0$$
$$\lambda^2 - 5\lambda + 4 = 0$$
$$(\lambda - 1)(\lambda - 4) = 0$$
$$\lambda_1 = 1$$
$$\lambda_2 = 4$$

Eigenvector for $\lambda_1 = 1$: solve $(A - I)\vec{k}_1 = \vec{0}$.

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $v_1 + 2v_2 = 0$, only constraint.

Let $v_2 = 1$, then $v_1 = -2$.

So $\vec{k}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 4$: solve $(A - 4I)\vec{k}_2 = \vec{0}$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2v_1 + 2v_2 &= 0 \\ v_1 - v_2 &= 0 \end{aligned}$$

$\Rightarrow$ $v_1 = v_2$, only constraint. Let $v_1 = 1$, then $v_2 = 1$.

So. $\vec{k}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

General soln.

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.}$$

(3.) $x' = -x + 2y$

$y' = -5x + y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(1-\lambda) + 10 = 0$$

$$-1 + \lambda^2 + 10 = 0$$

$$\lambda^2 + 9 = 0$$

$$\lambda^2 = -9$$

$$\lambda = \pm\sqrt{-9} = \pm 3i$$

Eigenvector for $\lambda = 3i$: solve $(A - 3i\, I)\vec{k}_1 = \vec{0}$

$$\begin{pmatrix} -1-3i & 2 \\ -5 & 1-3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -5v_1 + (1-3i)v_2 = 0$, only constraint

$\Rightarrow (1-3i)v_2 = 5v_1$

let $v_2 = 5$, then $v_1 = 1-3i$.

So $\vec{k}_1 = \begin{pmatrix} 1-3i \\ 5 \end{pmatrix}$.

Automatically know an eigenvector for $\lambda = -3i$: $\vec{k}_2 = \begin{pmatrix} 1+3i \\ 5 \end{pmatrix}$.

General soln in complex form:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1-3i \\ 5 \end{pmatrix} e^{3it} + c_2 \begin{pmatrix} 1+3i \\ 5 \end{pmatrix} e^{-3it}}$$

Real form:

$$\boxed{\vec{x}(t) = c_1 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 3t - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \sin 3t \right] + c_2 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 3t + \begin{pmatrix} -3 \\ 0 \end{pmatrix} \sin 3t \right]}$$

(4.)

$$x' = 4x + 5y$$
$$y' = -2x + 6y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for eigenvalues : $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(6-\lambda) + 10 = 0$$
$$24 - 10\lambda + \lambda^2 + 10 = 0$$
$$\lambda^2 - 10\lambda + 34 = 0$$

$$\lambda = \frac{10 \pm \sqrt{100 - 136}}{2} = \frac{10 \pm \sqrt{-36}}{2} = \frac{10 \pm 6i}{2} = 5 \pm 3i$$

Eigenvector for $\lambda = 5+3i$: solve $(A - (5+3i)I)\vec{k}_1 = \vec{0}$

$$\begin{pmatrix} 4-(5+3i) & 5 \\ -2 & 6-(5+3i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad -2v_1 + (1-3i)v_2 = 0$, only constraint

$\Rightarrow \quad (1-3i)v_2 = 2v_1$

Let $v_2 = 2$. Then $v_1 = 1-3i$.

So $\vec{k}_1 = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix}$

Eigenvector for $\lambda = 5 - 3i$: $\vec{k}_2 = \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$.

General soln in complex form:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} e^{(5+3i)t} + c_2 \begin{pmatrix} 1+3i \\ 2 \end{pmatrix} e^{(5-3i)t}}$$

In real form:

$$\boxed{\begin{aligned} \vec{x}(t) = c_1 &\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos 3t - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \sin 3t \right] e^{5t} \\ &+ c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos 3t + \begin{pmatrix} -3 \\ 0 \end{pmatrix} \sin 3t \right] e^{5t} \end{aligned}}$$

5.

$$x' = -8x - y$$
$$y' = 16x$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for eigenvalues : $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -8-\lambda & -1 \\ 16 & -\lambda \end{vmatrix} = 0$$

$$8\lambda + \lambda^2 + 16 = 0$$
$$(\lambda+4)^2 = 0$$
$$\lambda_1 = \lambda_2 = -4.$$

Find eigenvector: solve $(A + 4I)\vec{k} = \vec{0}$

$$\begin{pmatrix} -4 & -1 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $-4v_1 - v_2 = 0$, only constraint

$\Rightarrow$ $-4v_1 = v_2$

Let $v_1 = 1$, then $v_2 = -4$.

So $\vec{k} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Now we find $\vec{p}$ s.t. $(A+4I)\vec{p} = \vec{k}$.

$$\begin{pmatrix} -4 & -1 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$\Rightarrow \quad -4p_1 - p_2 = 1$, only constraint

$\Rightarrow \quad p_2 = -4p_1 - 1$

Let $p_1 = 1$, then $p_2 = -5$. So $\vec{p} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

General soln:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t} + c_2 \left[\begin{pmatrix} 1 \\ -4 \end{pmatrix} te^{-4t} + \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-4t} \right]}$$

6.

$$x' = x/2$$
$$y' = x + y/2 \qquad x(0) = 3, \quad y(0) = 5$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 & 0 \\ 1 & 1/2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1/2 - \lambda & 0 \\ 1 & 1/2 - \lambda \end{vmatrix} = 0$$

$$(1/2 - \lambda)^2 = 0$$

$$\lambda_1 = \lambda_2 = 1/2$$

Find eigenvector: solve $(A - \tfrac{1}{2} I)\vec{k} = \vec{0}$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad v_1 = 0$, only constraint. Let $v_2 = 1$.

So $\vec{k} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now find $\vec{p}$ s.t. $\left(A - \frac{1}{2}I\right)\vec{p} = \vec{k}$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow p_1 = 1$, only constraint. Let $p_2 = 0$

So $\vec{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

General soln:

$$\vec{x}(t) = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{1/2t} + c_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}te^{1/2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{1/2t}\right]$$

Plug in initial values:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} c_2 \\ c_1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} c_1 &= 5 \\ c_2 &= 3 \end{aligned}$$

Soln. to IVP:

$$\boxed{\vec{x}(t) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}e^{1/2t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}te^{1/2t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}e^{1/2t}}$$

(7.)

$$x' = 2x + 4y + 4z$$
$$y' = -x - 2y$$
$$z' = -x - 2z$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solve for eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 4 & 4 \\ -1 & -2-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$-\begin{vmatrix} 4 & 4 \\ -2-\lambda & 0 \end{vmatrix} - (2+\lambda)\begin{vmatrix} 2-\lambda & 4 \\ -1 & -2-\lambda \end{vmatrix} = 0$$

$$-4(2+\lambda) + (2+\lambda)^2(2-\lambda) - 4(2+\lambda) = 0$$
$$(2+\lambda)^2(2-\lambda) - 8(2+\lambda) = 0$$
$$(2+\lambda)\left[(2+\lambda)(2-\lambda) - 8\right] = 0$$
$$(2+\lambda)(\lambda^2+4) = 0$$

$$\Rightarrow \quad \lambda_1 = -2$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Eigenvector for $\lambda_1 = -2$: solve $(A+2I)\vec{k}_1 = \vec{0}$

$$\begin{pmatrix} 4 & 4 & 4 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad v_1 = 0$

$\cancel{4v_1} + 4v_2 + 4v_3 = 0 \Rightarrow v_2 = -v_3$

Let $v_3 = 1$. Then $v_2 = -1$.

So $\vec{k}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 2i$: solve $(A - 2i\,I)\vec{k}_2 = \vec{0}$

$$\left(\begin{array}{ccc|c} 2-2i & 4 & 4 & 0 \\ -1 & -2-2i & 0 & 0 \\ -1 & 0 & -2-2i & 0 \end{array}\right)$$

$$\Big\downarrow \; -1 \cdot R_2 \to R_2$$

$$\left(\begin{array}{ccc|c} 2-2i & 4 & 4 & 0 \\ 1 & 2+2i & 0 & 0 \\ -1 & 0 & -2-2i & 0 \end{array}\right)$$

$$\Big\downarrow \; \begin{array}{l} R_1 - (2-2i)R_2 \to R_1 \\ R_3 + R_2 \to R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & -4 & 4 & 0 \\ 1 & 2+2i & 0 & 0 \\ 0 & 2+2i & -2-2i & 0 \end{array}\right)$$

$$\Big\downarrow \; R_1 \rightleftarrows R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2+2i & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 2+2i & -2-2i & 0 \end{array}\right)$$

$$\Big\downarrow \; -\tfrac{1}{4}R_2 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2+2i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2+2i & -2-2i & 0 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_1 - (2+2i)R_2 \to R_1 \\ R_3 - (2+2i)R_2 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2+2i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left.\begin{array}{l} v_1 + (2+2i)\, v_3 = 0 \\ v_2 = v_3 \end{array}\right\}$$

Let $v_3 = 1$. Then $v_2 = 1$ and

$v_1 = -2-2i$

So $\vec{k}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix}$

Eigenvector for $\lambda_3 = -2i$ is $\vec{k}_3 = \begin{pmatrix} -2+2i \\ 1 \\ 1 \end{pmatrix}$.

General soln. in complex form

$$\vec{x}(t) = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix} e^{2it} + c_3 \begin{pmatrix} -2+2i \\ 1 \\ 1 \end{pmatrix} e^{-2it}$$

General soln in real form:

$$\vec{x}(t) = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \sin 2t \right]$$

$$+ \; c_3 \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \sin 2t \right]$$