Solutions of HW#10

1. Solution.

$$\begin{aligned} x+y-2z&=14 \\ 2x-y+z&=0 \\ 6x+3y+4z&=1 \end{aligned} \iff \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 6 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 1 \end{bmatrix} \qquad A\vec{X}=\vec{b}$$

Then we can write the augemented matrix as below

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 14 \\ 2 & -1 & 1 & 0 \\ 6 & 3 & 4 & 1 \end{array}\right] \xrightarrow[R_3-6\cdot R_1]{R_2-2\cdot R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 14 \\ 0 & -3 & 5 & -28 \\ 0 & -3 & 16 & -83 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 14 \\ 0 & -3 & 5 & -28 \\ 0 & 0 & 11 & -55 \end{array}\right]$$

Go back to the equations: $11\cdot z=-55$, $\Rightarrow z=-5$

$-3y+5z=-28 \Rightarrow y=1$; $x+y-2z=14 \Rightarrow x=3$

The solutions $\vec{X}=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$

2. Solution. Rearange the equations first!

$$\begin{aligned} x+y+z&=9 \\ 4x-3y+3z&=1 \\ 5x-2y+4z&=10 \end{aligned} \iff \begin{bmatrix} 1 & 1 & 1 \\ 4 & -3 & 3 \\ 5 & -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 4 & -3 & 3 & 1 \\ 5 & -2 & 4 & 10 \end{array}\right] \xrightarrow[R_3-5\cdot R_1]{R_2-4\cdot R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -7 & -1 & -35 \\ 0 & -7 & -1 & -35 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -7 & -1 & -35 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2^{nd} row: $-7\cdot y-z=-35$ $\qquad y=-\frac{1}{7}z+5$

1^{st} row: $1\cdot x+1\cdot y+1\cdot z=9 \Rightarrow x=-\frac{6}{7}z+4$

$$\therefore \vec{X}=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{6}{7}z+4 \\ -\frac{1}{7}z+5 \\ z \end{bmatrix} = z\begin{bmatrix} -\frac{6}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

3. Solution. Rearrange the equations first!

$$\begin{aligned} x-y-5z&=7 \\ 5x+4y-16z&=-10 \\ y+z&=-5 \end{aligned} \iff \begin{bmatrix} 1 & -1 & -5 \\ 5 & 4 & -16 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -5 & 7 \\ 5 & 4 & -16 & -10 \\ 0 & 1 & 1 & -5 \end{array}\right] \xrightarrow{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & -1 & -5 & 7 \\ 0 & 9 & 9 & -45 \\ 0 & 1 & 1 & -5 \end{array}\right] \xrightarrow{\frac{1}{9}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -5 & 7 \\ 0 & 1 & 1 & -5 \\ 0 & 1 & 1 & -5 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & -1 & -5 & 7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2^{nd} row: $y+z=-5 \Rightarrow y=-z-5$; 1^{st} row: $x-y-5z=7 \Rightarrow x=4z+2$

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4z+2 \\ -z-5 \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} z + \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$$

4. $\det A = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{vmatrix} = 4\cdot1\cdot0 + 2\cdot0\cdot(-1) + 3(2)(-2) - 3\cdot1\cdot(-1) - 2\cdot2\cdot0 - 4\cdot(-2)0$

$= -9 \neq 0 \quad \therefore \underline{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}_{3\times1}$

5. $\det A = \begin{vmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ 8 & 10 & -6 \end{vmatrix} = 0$

$A\underline{X} = \vec{0}$, $\left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 4 & 2 & -2 & 0 \\ 8 & 10 & -6 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_1 - R_2} \left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 4 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 0 & -6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

2^{nd} row: $-6y + 2z = 0 \quad \therefore y = \frac{1}{3}z$

1^{st} row: $2x + 4y - 2z = 0 \Rightarrow x = \frac{1}{3}z$

$$\therefore \underline{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3}z \\ \frac{1}{3}z \\ z \end{pmatrix} = z\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

6. $\det A = \begin{vmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -1\cdot(-1)^{1+1}\cdot\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} + 3\cdot(-1)^{1+2}\cdot\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0\cdot(-1)^{1+3}\cdot\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$

$= -[-4-1] - 3\cdot2 + 0 = -1 \neq 0 \qquad A\underline{X} = \vec{0}$

$\left(\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$\xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 - 3R_2} \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \therefore \begin{array}{l} -x = 0 \\ y = 0 \\ z = 0 \end{array} \Rightarrow \underline{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

7. $A = \begin{pmatrix} -1 & 2 \\ -7 & -8 \end{pmatrix}$, $\det(A - \lambda I) = 0$, $\begin{vmatrix} -1-\lambda & 2 \\ -7 & -8-\lambda \end{vmatrix} = (\lambda+1)(\lambda+8) + 14$

$= \lambda^2 + 9\lambda + 22 = 0$

$\therefore \lambda = -\frac{9}{2} \pm i\frac{\sqrt{7}}{2}$. $\lambda_1 = -\frac{9}{2} + i\frac{\sqrt{7}}{2}$, we have $(A - \lambda_1 I)\vec{V}_1 = 0$

$\begin{pmatrix} \frac{7}{2} - i\frac{\sqrt{7}}{2} & 2 \\ -7 & -\frac{7}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$ $\qquad -7k_1 + (-\frac{7}{2} - i\frac{\sqrt{7}}{2})k_2 = 0$

$k_1 = (-\frac{1}{2} - i\frac{1}{2\sqrt{7}})k_2$

$\therefore \vec{V}_1 = \begin{pmatrix} -\frac{1}{2} - i\frac{1}{2\sqrt{7}} \\ 1 \end{pmatrix} k_2$, k_2 is an arbitrary constant.