Which method do you choose?

① $(1+x)\,dy - y\,dx = 0$ → First-order separation of variables

② $\frac{dy}{dx} = \sin(5x)\, y$ → First-order separation of variables

③ $y' + 4y = f(t)$ where $y(0) = 1$ Laplace Transform

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ 1 & t \ge 4 \end{cases}$$

④ $x^3 y'' + 4x^2 y' + 3y = 0$ 2nd-order non-constant coeff. homogeneous. } power series

⑤ $y'' + y' - 6y = 2x$ 2nd-order constant-coeff non-homog. } judicial guessing ↳ (polynomial)

⑥ $\frac{dP}{dt} = P - P^2$ 1st-order separation of variables

⑦ $y'' + 2y' + 2y = e^x \cos(2x)$ 2nd-order constant coeff. non-homog } judicial guessing

⑧ $(5y - 2x)\,dy - 2y\,dx = 0$ 1st-order exact eqn

⑨ $\frac{dy}{dx} + x^6 e^x = 4y$ 1st order integrating factor

⑩ $y'' - y' + \frac{1}{4} y = 3 + e^x$ 2nd-order constant coeff non-homog. } judicial guessing

⑪ $(x^3 + 4x)\, y'' - 2xy' + 6y = 0$ 2nd-order, nonconstant coeff. homog. ↳ Power series.

⑫ $\frac{dx}{dt} = 3x - 18y$; $\frac{dy}{dt} = 2x - 9y$ → systems of eqns.

Extra Problems

① $$\frac{dx}{dt} = 3x - y$$
$$\frac{dy}{dt} = 9x - 3y$$

② $y'' + 5y' + 2y = 4\,\delta(t - 2\pi) \qquad y'(0) = y(0) = 0.$

③ $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = t^2 e^{-t}$

④ $X' = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} X$

⑤ $y' - \frac{11}{9}x = y$

⑥ $(x^2+1)y' + 2xy = 0.$

⑦ $$\frac{dx}{dt} = x + 2y$$
$$\frac{dx}{dt} = 4x + 3y.$$

⑧ $\dfrac{dy}{dx} + 2xy^2 = 0, \quad y(0) = 1.$

⑨ $4y'' + 9y = 15$ using variation of parameters.

Soln to Extra Problems.

① $\dfrac{dx}{dt} = 3x - y$

$\dfrac{dy}{dt} = 9x - 3y.$ $\longrightarrow X' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} X$

$\det(A - \lambda I) = 0.$

$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0.$

$(3-\lambda)(-3-\lambda) + 9 = 0$

$-9 - 3\lambda + 3\lambda + \lambda^2 + 9 = 0.$

$\lambda^2 = 0.$

$\lambda = 0.$

$\underline{\lambda = 0}$

$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$

$\boxed{v_1 = \frac{1}{3} v_2}$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

First solution:

$\vec{X}_1(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t}$

$\underline{\text{Second soln}}$ $(A - \lambda I)\vec{V}_2 = \vec{V}_1$

$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right]$

$3R_1 - R_2 \left[\begin{array}{cc|c} 3 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\boxed{v_1 = \frac{1 + v_2}{3}}$

$\vec{V}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{X}_2(t) = \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) e^{0t}.$

$\boxed{\vec{X}(t) = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)}$

(2')

$$y'' + 5y' + 2y = 4\,\delta(t-2\pi) \qquad y'(0) = y(0) = 0.$$

$$s^2 Y(s) - \cancel{s Y'(0)} - \cancel{Y(0)} + 5sY(s) + 2Y(s) = \mathcal{L}\{4\,\delta(t-2\pi)\}$$

$$Y(s)\left(s^2+5s+2\right) = 4\left(e^{-2\pi s}\right)$$

$$\frac{25}{\frac{-8}{17}}$$

$$Y(s) = \frac{4e^{-2\pi s}}{s^2+5s+2}.$$

complete the square

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{\left(s+\frac{5}{2}\right)^2 + 2 - \left(\frac{5}{2}\right)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{\left(s+\frac{5}{2}\right)^2 - \frac{17}{4}}\right\}$$

$$= 4\,\mathcal{U}(t-2\pi)\left[\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^2 - \frac{17}{4}}\right\}\right]_{t\to t-2\pi}$$

$$= 4\,\mathcal{U}(t-2\pi)\left[\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2-\frac{17}{4}}\right|_{s\to s+\frac{5}{2}}\right\}\right]_{t\to t-2\pi.}$$

$$= 4\,\mathcal{U}(t-2\pi)\left[\sqrt{\frac{4}{17}}\,e^{-\frac{5}{2}t}\sinh\left(\sqrt{\frac{17}{4}}\,t\right)\right]_{t\to t-2\pi}$$

$$\boxed{y(t) = 4\,\mathcal{U}(t-2\pi)\sqrt{\frac{4}{17}}\,e^{-\frac{5}{2}(t-2\pi)}\sinh\left(\sqrt{\frac{17}{4}}(t-2\pi)\right)}$$

(3)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t^2 e^{-t}$$

$$y'' + 3y' + 2 = 0.$$

$$y(t) = e^{\lambda t}$$

$$\lambda^2 + 3\lambda + 2 = 0,$$

$$(\lambda+1)(\lambda+2)$$

$$\boxed{y(t) = c_1 e^{-t} + c_2 e^{-2t}}$$

$$y_p = ((At^2 + Bt + C)e^{-t})t$$

$$y_p = (At^3 + Bt^2 + Ct)e^{-t}$$

$$y_p' = (A3t^2 + 2Bt + C)e^{-t} + -e^{-t}(At^3 + Bt^2 + Ct)$$

$$y_p'' = (6At + 2B)e^{-t} + -e^{-t}(A3t^2 + 2Bt + C) - (3At^2 + 2Bt + C) \cdot e^{-t} + y_p.$$

$\cancel{(6At + 2B)e^{-t}}$.

$$(6At + 2B)e^{-t} - 2(A3t^2 + 2Bt + C)e^{-t} + y_p + 3(A3t^2 + 2Bt + C)e^{-t}$$

$$-3y_p + 2y_p = t^2 e^{-t}$$

$t^3 e^{-t}|$ $\quad 0 = 0.$

$t^2 e^{-t}|$ $\quad -6A + 9A = 1 \rightarrow 3A = 1 \quad \boxed{A = \frac{1}{3}}$

$te^{-t}|$ $\quad 6A - 4B + 6B = 0. \rightarrow 2 + 2B = 0. \rightarrow \boxed{B = -1}$

$e^{-t}|$ $\quad 2B - 2C + 3C = 0 \rightarrow 2(-1) + C = 0. \quad \boxed{C = 2}$

$$\boxed{y_{gen'l}(t) = c_1 e^{-t} + c_2 e^{-2t} + \left(\tfrac{1}{3}t^3 - t^2 + 2t\right)e^{-t}}$$

④

$$X' = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} X$$

$$(1-\lambda)(-3-\lambda)+8=0.$$

$$-3-\lambda+3\lambda+\lambda^2+8=0.$$

$$\lambda^2+2\lambda+5=0.$$

$$\lambda = \frac{-2 \pm \sqrt{4-4(5)}}{2} \qquad \frac{\sqrt{16}^{\,4}}{2}$$

$$\boxed{\lambda = -1 \pm 2i}$$

$$\begin{bmatrix} 1+1\mp 2i & -8 \\ 1 & -3+1\mp 2i \end{bmatrix} \vec{V} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2\mp 2i & -8 & 0 \\ 1 & -2\mp 2i & 0 \end{array}\right]$$

$$R_1-(2\mp 2i)R_2 \left[\begin{array}{cc|c} 2\mp 2i & -8 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad 8-(2\mp 2i)(-2\mp 2i)$$

$$8-(-4\mp 4i\pm 4i-4)=0.$$

$$\boxed{V_1 = \frac{-8V_2}{2\mp 2i}} \longrightarrow \vec{V} = \begin{bmatrix} -8 \\ 2\mp 2i \end{bmatrix}$$

$$\vec{X}(t) = C_1 \begin{bmatrix} -8 \\ 2-2i \end{bmatrix} e^{(-1+2i)t} + C_2 \begin{bmatrix} -8 \\ 2+2i \end{bmatrix} e^{(-1-2i)t}$$

$$\vec{X}_1 = \left[\begin{bmatrix} -8 \\ 2 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin(2t)\right] e^{-t}$$

$$\vec{X}_2 = \left[\begin{bmatrix} 0 \\ -2 \end{bmatrix} \cos(2t) + \begin{bmatrix} -8 \\ 2 \end{bmatrix} \sin(2t)\right] e^{-t}$$

$$\boxed{\vec{X}_g(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)}$$

5

$$\frac{dy}{dx} - \frac{11}{9}x^{1} = y$$

$$\frac{dy}{dx} - y = \frac{11}{9}x^{1}$$

$$\mu(x) = e^{-\int 1\,dx} = e^{-x}$$

$$e^{-x}\left(\frac{dy}{dx} - y = \frac{11}{9}x^{1}\right)$$

$$\frac{d}{dx}\left(e^{-x}\cdot y\right) = \frac{11}{9}x^{1}e^{-x}$$

$$\int \frac{d}{dx}\left(e^{-x}y\right)dx = \int \frac{11}{9}x^{1}e^{-x}\,dx.$$

$u = x \qquad v = -e^{-x}$

$du = dx \qquad dv = e^{-x}\,dx$

$-x(e^{-x}) - \int(-e^{-x})\,dx.$

$-xe^{-x} - e^{-x} + C$

$$\cancel{y = \left(\frac{11}{9}x^{2}e^{-x}\right)e^{x}}$$

$$e^{-x}y = \cancel{\frac{11}{9}(-x)(e} \; \frac{11}{9}(e^{-x})(-x-1).$$

$$\cancel{y(x) = \frac{11}{9}x^{2}}$$

$$\boxed{y(x) = -\frac{11}{9}(x+1) + Ce^{x}.}$$

(6) $(x^2+1)y' + 2xy = 0$.

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad\qquad (x^2+1)x^{n-1} = x^{n+1} + x^{n-1}$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \qquad\qquad 2x(x^n) = 2x^{n+1}$$

$$\sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n 2x^{n+1} = 0.$$

$\boxed{\begin{matrix} k=n+1 \\ n=k-1 \end{matrix}}$ $\qquad$ $\boxed{\begin{matrix} k=n-1 \\ n=k+1 \end{matrix}}$ $\qquad$ $\boxed{\begin{matrix} k=n+1 \\ n=k-1 \end{matrix}}$

$$\sum_{k=2}^{\infty} c_{k-1}(k-1)x^k + \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + \sum_{k=1}^{\infty} c_{k-1}2x^k = 0.$$

$$\sum_{k=2}^{\infty}\left[c_{k-1}(k-1) + c_{k+1}(k+1) + 2c_{k-1}\right]x^k + c_1x^0 + c_2 2x^1 + c_0 2x^1 = 0$$

$x^0|$ $\quad c_1 = 0$

$x^1|$ $\quad 2c_2 + 2c_0 \to c_2 = -c_0$

$k \geq 2|$

$$c_{k+1} = \frac{(-2-k+1)c_{k-1}}{k+1} = \left(\frac{-(2+k)+1}{k+1}\right)c_{k-1}$$

$$= \left(-1 + \frac{-1+1}{k+1}\right)c_{k-1}$$

$$\boxed{c_{k+1} = -c_{k-1}}$$

k	c_{k+1}
2	$c_3 = -c_1 = 0.$
3	$c_4 = -c_2 = -(-c_0) = c_0.$
4	$c_5 = -c_3 = 0.$
5	$c_6 = -c_4 = -c_0.$

$$y(x) = c_0 + -c_0 x^2 + c_0 x^4 - c_0 x^6 + \dots$$

$$y(x) = c_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{1}$$

$$\boxed{y(x) = c_0 \sum_{k=0}^{\infty} (-1)^k x^{2k}.}$$

7)

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 4x + 3y$$

$$\longrightarrow \quad X' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} X$$

find $\lambda, \vec{v}$ s.t. $A\vec{v} = \lambda\vec{v}$.

$$\det(A - \lambda I) = 0.$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(3-\lambda) - 8 = 0.$$

$$3 - 4\lambda + \lambda^2 - 8 = 0.$$

$$\lambda^2 - 4\lambda - 5 = 0.$$

$$(\lambda + 1)(\lambda - 5) = 0.$$

$$\lambda = -1, 5$$

$\underline{\lambda = -1}$

$$(A - \lambda I)\vec{v} = 0.$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\boxed{v_1 = -v_2}$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 5}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$\boxed{v_1 = \tfrac{1}{2} v_2}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}}$$

⑧

$$\frac{dy}{dx} + 2xy^2 = 0. \qquad y(0) = 1$$

$$\frac{dy}{dx} = -2xy^2$$

$$\frac{1}{y^2}\frac{dy}{dx} = -2x$$

$$\frac{d}{dx}\left(-\frac{1}{y}\right) = -2x$$

$$\int \frac{d}{dx}\left(-\frac{1}{y}\right)dx = -2\int x\,dx.$$

$$-\frac{1}{y} = -2\left(\frac{x^2}{2} + c\right)$$

$$\boxed{\frac{+1}{2\left(\frac{x^2}{2}+c\right)} = y(x)}$$

rearrange...

$$\boxed{y(x) = \frac{+1}{x^2 + 2c}}$$

Final answer.

$$\boxed{y(x) = \frac{-1}{x^2+1}}$$

$$y(0) = \frac{+1}{0^2 + 2c} = 1 \longrightarrow +1 = 2c$$

$$\boxed{c = +\frac{1}{2}}$$

$4y'' + 9y = 15$ w/ variation of parameters.

$4y_c'' + 9y_c = 0$.

ansatz $y_c(x) = e^{\lambda x}$

$4\lambda^2 + 9 = 0$

$\lambda^2 = -\frac{9}{4}$

$\lambda = \pm \frac{3}{2} i$

$y_c = c_1 e^{\frac{3}{2} i x} + c_2 e^{-\frac{3}{2} i x}$

or

$y_c = \tilde{c}_1 \cos\left(\frac{3}{2}x\right) + \tilde{c}_2 \sin\left(\frac{3}{2}x\right)$

$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$

$f(x) = \frac{15}{4}$ $\quad W = \begin{vmatrix} \cos\frac{3}{2}x & \sin\frac{3}{2}x \\ -\frac{3}{2}\sin\frac{3}{2}x & \frac{3}{2}\cos\frac{3}{2}x \end{vmatrix} = \frac{3}{2}\left(\cos\left(\frac{3}{2}x\right)\right)^2 + \frac{3}{2}\left(\sin\left(\frac{3}{2}x\right)\right)^2 = \frac{3}{2}$

$u_1'(x) = \frac{-y_2 f(x)}{W} = \frac{-\sin\frac{3}{2}x \; \frac{15}{4}}{\frac{3}{2}} = -\frac{15}{4}\frac{2}{3}\sin\frac{3}{2}x$

$= -\frac{5}{2}\sin\frac{3}{2}x$

$u_1(x) = -\frac{5}{2}\int \sin\frac{3}{2}x \, dx = -\frac{5}{2}\frac{2}{3}\left(-\cos\frac{3}{2}x\right) = \frac{5}{3}\cos\frac{3}{2}x.$

$$u_2'(x) = \frac{+y_1 f(x)}{W} = \frac{\cos\frac{3}{2}x \ \frac{15}{4}}{\frac{3}{2}} = \frac{15}{4}\frac{2}{3}\cos\frac{3}{2}x = \frac{5}{2}\cos\frac{3}{2}x$$

$$u_2(x) = \frac{5}{2}\int \cos\frac{3}{2}x\,dx = \frac{5}{2}\frac{2}{3}\sin\frac{3}{2}x = \frac{5}{3}\sin\frac{3}{2}x$$

$$y_p = \left(\frac{5}{3}\cos\frac{3}{2}x\right)\cos\left(\frac{3}{2}x\right) + \left(\frac{5}{3}\sin\left(\frac{3}{2}x\right)\right)\left(\sin\frac{3}{2}x\right)$$

$$y_p = \frac{5}{3}$$

$$y_g(x) = \tilde{c}_1\cos\left(\frac{3}{2}x\right) + \tilde{c}_2\sin\left(\frac{3}{2}x\right) + \frac{5}{3}$$