Problem 3. (30 pts) Find the general solution of the differential equation. An implicit solution is fine.

$$2(y^2 - e^{-x}\sin 2y)\frac{dy}{dx} = e^{-x}\cos 2y$$

Proper form:

$$\underbrace{-e^{-x}\cos 2y}_{M} + \underbrace{2(y^2 - e^{-x}\sin 2y)}_{N}\frac{dy}{dx} = 0$$

check $\boxed{\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x} \text{ to see if } M = \frac{\partial f}{\partial x} \text{ and if } N = \frac{\partial f}{\partial y}.}$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[-e^{-x}\cos 2y\right] = -e^{-x}(-\sin 2y \cdot 2) = +2\sin 2y\, e^{-x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[2(y^2 - e^{-x}\sin 2y)\right] = 2(-\sin 2y(-e^{-x})) = +2\sin 2y\, e^{-x}$$

✓ proceed.

Now, we know that $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$.

① $\boxed{M = \frac{\partial f}{\partial x}}$

$$-e^{-x}\cos 2y = \frac{\partial f}{\partial x}$$

$$\text{“}\int\text{''} -e^{-x}\cos 2y\, dx = \text{“}\int\text{''} \frac{\partial}{\partial x}(f)\,dx$$

$$-\cos 2y \frac{e^{-x}}{-1} + g(y) = f$$

$$\boxed{\cos 2y\, e^{-x} + g(y) = f}$$

② $\boxed{N = \frac{\partial f}{\partial y}}$

$$2(y^2 - e^{-x}\sin 2y) = \frac{\partial f}{\partial y}$$

$$2(y^2 - e^{-x}\sin 2y) = \frac{\partial}{\partial y}\left[\cos(2y)\, e^{-x} + g(y)\right]$$

$$2y^2 - 2e^{-x}\sin 2y = e^{-x}(-\sin 2y \cdot 2) + g'(y)$$

$$2y^2 = g'(y)$$

$$\int 2y^2\, dy = \int \frac{d}{dy} g(y)\, dy$$

$$\boxed{\frac{2y^3}{3} = g(y)}$$

$\therefore$ $\boxed{\cos(2y)e^{-x} + \frac{2}{3}y^3 = C}$