Exam 1 Practice

1) $\underbrace{(x^3+y^3)}\,dx + 3xy^2\,dy = 0.$

A sum of a f(x) & f(y). Irreducible. Tells me I can't use separation of variables.

$\frac{dy}{dx} = f(x)\,g(y).$

Is it exact?

$M = x^3+y^3$

$N = 3xy^2$

$M_y \stackrel{?}{=} N_x$

x const | y const.

$3y^2$ | $3y^2 \cdot 1$

It is exact.

① $M = \frac{\partial f}{\partial x}$

$x^3+y^3 = \frac{\partial f}{\partial x}$

"$\int$" $x^3+y^3\,dx$ = "$\int$" $\frac{\partial}{\partial x} f\,dx.$

y const.

$\boxed{\frac{x^4}{4} + y^3x + g(y) = f.}$

② $N = \frac{\partial f}{\partial y}$

$3xy^2 = \frac{\partial}{\partial y}\left[\frac{x^4}{4} + y^3x + g(y)\right]$

x const.

$3xy^2 = x3y^2 + g'(y).$

$0 = g'(y)$

$\boxed{0 = g(y)}$

③ $f(x,y) = \text{Constant}.$

$\therefore \boxed{\frac{x^4}{4} + y^3x = C.}$

2) $\underbrace{\frac{dr}{d\theta} + r\sec\theta}_{} = \cos\theta.$

Sum of First deriv of r and r leads to Linear Eqn.

Does it work? yes.

$$\frac{dy}{dx} + \underbrace{P(x)}_{\downarrow} y = \underbrace{Q(x)}_{\downarrow}$$

$$\frac{dr}{d\theta} + \sec\theta\, r = \cos\theta.$$

Integrating factor.

$$\mu(\theta) = e^{\int P(\theta)\,d\theta} = e^{\int \sec\theta\, d\theta} = e^{\int \sec\theta\left(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\right)d\theta}$$

$$= e^{\ln|\sec\theta+\tan\theta|} \quad \text{via u-sub.}$$

$$= |\sec\theta + \tan\theta|$$

$\sec(\theta) \to \infty$ as $\theta \to -\frac{\pi}{2}, \frac{\pi}{2}$

$\tan\theta \to \infty$ as $\theta \to \frac{\pi}{2}$

$\tan\theta \to -\infty$ as $\theta \to -\frac{\pi}{2}$

$$\boxed{\mu(\theta) = \sec\theta + \tan\theta \quad \text{for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

$$\underbrace{(\sec\theta + \tan\theta)\left(\frac{dr}{d\theta} + r\sec\theta\right)}_{\text{result of product rule}} = (\sec\theta + \tan\theta)\cos\theta.$$

$$\frac{d}{d\theta}\left((\sec\theta+\tan\theta)r\right) = 1 + \sin\theta$$

$$\int \frac{d}{d\theta}\left((\sec\theta+\tan\theta)r\right)d\theta = \int 1 + \sin\theta\, d\theta$$

$$(\sec\theta+\tan\theta)r = \theta - \cos\theta + C$$

$$\therefore \boxed{r(\theta) = \frac{\theta - \cos\theta + C}{\sec\theta + \tan\theta}}$$

3) $x^2 y' + x(x+2)y = e^x$

Indicates that Linear Eqn method works

$$\frac{dy}{dx} + \underbrace{\frac{x^2+2x}{x^2}}\, y = e^x x^{-2}$$

$$P(x) = 1 + \frac{2}{x} \longrightarrow \boxed{x \in (0,\infty)}$$

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int 1+\frac{2}{x}\,dx} = e^{x+2\ln|x|} = e^x e^{\ln|x^2|} = e^x x^2$$

$$\underbrace{e^x x^2\left(\frac{dy}{dx} + \left(1+\frac{2}{x}\right)y\right.} \left. = e^x x^{-2}\right)$$

result of the
product rule of

$$\frac{d\left(e^x x^2 \cdot y\right)}{dx} = e^{2x}$$

$$\int \frac{d}{dx}\left(e^x x^2 y\right)dx = \int e^{2x}\,dx.$$

$$e^x x^2 y = \frac{1}{2}e^{2x} + C$$

$$y(x) = \frac{1}{2}e^{2x}e^{-x}x^{-2} + Ce^{-x}x^{-2}$$

$$\therefore \boxed{y(x) = \frac{1}{2}e^x x^{-2} + Ce^{-x}x^{-2}}$$

4) $y\,dx = \underbrace{(ye^y - 2x)}\,dy$

Sum of a f(x) and f(y) that is irreducible → not separable.

Is it Exact? $M = y$ $\quad M_y = 1$

$N = -ye^y - 2x$ $\quad N_x = -2$. Nope.

Linear. $y\,dx = (ye^y - 2x)\,dy$ Divide Eqn by $y\,dx$

$1 = \left(e^y - \frac{2x}{y}\right)\frac{dy}{dx}$ Force coeff of $\frac{dy}{dx}$ to be 1

$\frac{-y}{2x} = \frac{-y}{2x}e^y + \frac{dy}{dx}$. Write in proper form?

$$\frac{dy}{dx} + y\underbrace{\left(\frac{1}{2x} - e^y\frac{1}{2x}\right)}_{P(x,y)} = 0.$$

Doesn't work!

Instead. $y\,dx = (ye^y - 2x)\,dy$ Divide Eqn by dy & y.

$\frac{dx}{dy} = \left(e^y - \frac{2x}{y}\right)$ Bring $-\frac{2x}{y}$ to other side

$$\boxed{\frac{dx}{dy} + \underbrace{\frac{2}{y}}_{P(y)}x = e^y} \qquad \boxed{y \in (0,\infty)}$$

Linear for $\frac{dx}{dy} \to x(y)$ soln

$\mu(y) = e^{\int P(y)dy} = e^{\int \frac{2}{y}dy} = e^{2\ln|y|} = e^{\ln y^2} = y^2$

$y^2\left(\frac{dx}{dy} + \frac{2}{y}x\right) = y^2e^y$

$\frac{d}{dy}(y^2 \cdot x) = y^2e^y$

$\int \frac{d}{dy}(y^2x)\,dy = \int y^2e^y\,dy$.

$y^2x = \int y^2e^y\,dy$

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4) continued

$$y^2x = \int y^2e^y\,dy.$$

IBP $\quad u = y^2 \quad v = e^y$

$du = 2y\,dy \quad dv = e^y\,dy$

$uv - \int v\,du$

$$y^2e^y - \int e^y\,2y\,dy$$

IBP. $\quad u = y \quad v = e^y$

$du = dy \quad dv = e^y\,dy$

$$y^2e^y - \left(ye^y - \int e^y\,dy\right)$$

$$y^2x = y^2e^y - ye^y + e^y + C$$

$$\boxed{x(y) = e^y - y^{-1}e^y + e^y y^{-2} + Cy^{-2}}$$

5) $\underbrace{\frac{dN}{dt} + N = Nte^{t+2}}$

Looks like Linear Eqn form. Check $P(x)$, $Q(x)$.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dN}{dt} + 1 \cdot N = \underbrace{Nte^{t+2}}_{Q(t,N)}$$

Not Linear!

Move $+N$ to RHS and you'll see that this problem is separable variables.

$$\frac{dN}{dt} = -N + Nte^{t+2}$$

$$\boxed{\frac{dN}{dt} = N\left(te^{t+2} - 1\right)}$$ Divide by N.

$$\frac{1}{N}\frac{dN}{dt} = \left(te^{t+2} - 1\right)$$ LHS is the derivative of $\ln|N|$

$$\frac{d}{dt}\left(\ln|N|\right) = te^{t+2} - 1$$ Int wrt t.

$$\int \frac{d}{dt}\left(\ln|N|\right)dt = \int\left(te^{t+2} - 1\right)dt$$

$$\ln|N| = \underbrace{\int te^{t+2}\,dt}_{\text{IBP}} - t + C$$

IBP

$u = t \qquad v = e^{t+2}$

$du = dt \qquad dv = e^{t+2}\,dt$

$$te^{t+2} - \int e^{t+2}\,dt$$

$$\ln|N| = te^{t+2} - e^{t+2} - t + C.$$ Exponentiate.

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$$e^{\ln|N|} = e^{te^{t+2} - e^{t+2} - t + C}$$

$$N = \pm e^{te^{t+2}} \cdot \frac{1}{e^{e^{t+2}} e^{t}} \cdot e^{C}$$

$$\therefore \boxed{N(t) = \frac{C e^{te^{t+2}}}{e^{e^{t+2}} e^{t}}}$$

OR

$$e^{(e^{t+2})(t-1)} \quad e^{-t} \quad e^{C}$$

$$N = \pm C \frac{e^{e^{t+2}(t-1)}}{e^{t}}$$

6) $x\frac{dy}{dx} + (3x+1)y = e^{-3x}$

Linear. Check $P(x)$, $Q(x)$, ok!

$$\frac{dy}{dx} + \left(3 + \frac{1}{x}\right)y = e^{-3x}x^{-1}$$

$$\boxed{x \in (0, \infty)}$$

$$\mu(x) = e^{\int P(x)dx} = e^{\int 3 + \frac{1}{x}\,dx} = e^{3x + \ln|x|} = xe^{3x}.$$

$$\underbrace{xe^{3x}\left(\frac{dy}{dx} + \left(3+\frac{1}{x}\right)y\right.}_{\text{Result of product rule}} \left. = e^{-3x}x^{-1}\right)$$

$$\frac{d}{dx}\left(xe^{3x}y\right) = 1$$

$$\int \frac{d}{dx}\left(xe^{3x}y\right)dx = \int 1\,dx$$

$$xe^{3x}y = x + C$$

$$\boxed{y(x) = e^{-3x} + Ce^{-3x}x^{-1}}$$

7) $x^2 \frac{dy}{dx} = \underbrace{y - xy}$, $y(-1) = -1$

difference of functions of x and y.
But I can factor them into $f(x)g(y)$.

$y - xy = y(1-x)$

$x^2 \frac{dy}{dx} = y(1-x)$ Divide by x^2

$\frac{dy}{dx} = y\frac{(1-x)}{x^2}$ Separable Variables

$\frac{1}{y}\frac{dy}{dx} = \frac{1-x}{x^2}$ LHS, recognize the derivative

$\frac{d}{dx}(\ln|y|) = \frac{1-x}{x^2}$ Integrate wrt x.

$\int \frac{d}{dx}(\ln|y|) = \int \frac{1-x}{x^2}\,dx \rightarrow x \neq 0$ $\boxed{x \in (0, -\infty)}$ ($-\infty$ because of IC $\boxed{y(-1) = -1}$)

$\ln|y| = \int \frac{1}{x^2} - \frac{1}{x}\,dx.$

$\ln|y| = -x^{-1} - \ln|x| + C$ Exponentiate.

$e^{\ln|y|} = e^{-x^{-1} - \ln|x| + C}$

$|y| = e^{-\frac{1}{x}} e^{\ln\left(\frac{1}{|x|}\right)} e^{C}$

abso. value absorbed into constant

$\boxed{\therefore y = Ce^{-\frac{1}{x}}\left(\frac{1}{-x}\right)} \rightarrow$ $-x$ because $x \in (0, -\infty)$.

Don't Forget $y(-1) = -1$

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$$y(x) = Ce^{-\frac{1}{x}}\left(-\frac{1}{x}\right)$$

can absorb negative sign into constant.

$$\boxed{y(x) = Ce^{-\frac{1}{x}}\frac{1}{x} \qquad x \in (0, -\infty)}$$

$$y(-1) = Ce^{1}\frac{1}{-1} = -1$$

$$-Ce = -1$$

$$C = \frac{1}{e} = e^{-1}$$

$$\boxed{y(x) = \frac{e^{-1-\frac{1}{x}}}{x}}$$

8) $$\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$$

ratio of fcns of x and y. Can you factor out into $f(x)g(y)$? Yes!

$$\frac{dy}{dx} = \frac{y(x+2)-(x+2)}{y(x-3)+(x-3)}$$

$$\boxed{\frac{dy}{dx} = \frac{(y-1)(x+2)}{(y+1)(x-3)}}$$

Mult by $\frac{y+1}{y-1}$

$$\frac{y+1}{y-1}\frac{dy}{dx} = \frac{x+2}{x-3}$$

$$\int \frac{y+1}{y-1}\,dy.$$

Because power of y in numerator & denominator are the same, use long division to rewrite

$\frac{y+1}{y-1}$ as $\frac{y-1}{y-1} + \frac{2}{y-1}$

$$\int 1 + \frac{2}{y-1}\,dy$$

LHS is. $\frac{d}{dx}\left(y + 2\ln|y-1|\right)$

$$\frac{d}{dx}\left(y+2\ln|y-1|\right) = \frac{x+2}{x-3}$$

Int wrt x.

$$\int \frac{d}{dx}\left(y+2\ln|y-1|\right)dx = \int \frac{x+2}{x-3}\,dx.$$

$$y+2\ln|y-1| = \int \frac{x-3}{x-3} + \frac{5}{x-3}\,dx$$

$$\boxed{y+2\ln|y-1| = x + 5\ln|x-3| + C}$$

exponentiated...

$$e^y(y-1)^2 = e^x(x-3)^5 C$$

9) $\underbrace{y\ln(x)\,dx}\; = \underbrace{\left(\frac{y+1}{x}\right)^2 dy}$

functions of x and y multiplied together.

Indicates separation of variables!

$$\boxed{\frac{y\ln(x)\,x^2}{(y+1)^2} = \frac{dy}{dx}.}$$

$$\ln(x)\,x^2 = \frac{(y+1)^2}{y}\frac{dy}{dx}$$

$$\int \frac{(y+1)^2}{y}\,dy$$

$$\int \frac{y^2+2y+1}{y}\,dy$$

$$\int y + 2 + \frac{1}{y}\,dy$$

RHS is

$$\ln(x)\,x^2 = \frac{d}{dx}\left(\frac{y^2}{2} + 2y + \ln|y|\right) \qquad \text{Int wrt } x.$$

$$\int x^2\ln(x)\,dx = \int \frac{d}{dx}\left(\frac{y^2}{2} + 2y + \ln|y|\right)dx.$$

IBP $u = \ln(x)$ $\quad v = x^3/3$

$du = \frac{1}{x}\,dx$ $\quad dv = x^2\,dx$

$uv - \int v\,du$

$$\frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx \rightarrow \boxed{\frac{x^3}{3}\ln(x) - \frac{x^3}{9} = \frac{y^2}{2} + 2y + \ln|y| + C}$$