math 527 HW 8 solutions

Problem 1

$$\begin{cases} x_2 - 3x_3 = 5 \\ 2x_1 + 3x_2 - x_3 = 7 \\ 4x_1 + 5x_2 - 2x_3 = 10 \end{cases}$$

In the Augmented matrix notation, this system becomes:

$$\left(\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right) \quad R_1 \longleftrightarrow R_3 \quad \text{Exchange } R_1 \,\&\, R_3 \text{ (Row 1 and Row 3)}$$

$$\left(\begin{array}{ccc|c} 4 & 5 & -2 & 10 \\ 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \end{array}\right) \quad R_1/4 \longrightarrow R_1 \quad \text{(Divide row 1 by 4)}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 5/4 & -1/2 & 5/2 \\ 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \end{array}\right) \quad -2R_1 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 5/4 & -1/2 & 5/2 \\ 0 & 1/2 & 0 & 2 \\ 0 & 1 & -3 & -5 \end{array}\right) \quad 2R_2 \longrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 5/4 & -1/2 & 5/2 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 1 & -3 & -5 \end{array}\right) \quad -R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 5/4 & -1/2 & 5/2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right) \quad -1/3\, R_3 \longrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 5/4 & -1/2 & 5/2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right) \quad \text{Done!}$$

now, starting from the last row of the final augmented matrix

we find:

$x_3 = 3$.

from the second row we get:

$x_2 = 4$.

from the first row we have:

$x_1 + 5/4\, x_2 - 1/2\, x_3 = 5/2 \Rightarrow$

$x_1 = -5/4(4) + 1/2(3) = -1 \Rightarrow \boxed{x_1 = 1}$

note that the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 5/4 & -1/2 & 5/2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

represents the following system of equations:

$$\begin{aligned} 1x_1 + 5/4\, x_2 - 1/2\, x_3 &= 5/2 \\ 0x_1 + 1x_2 + 0x_3 &= 4 \\ 0x_1 + 0x_2 + 1x_3 &= 3 \end{aligned} \quad \text{i.e} \quad \begin{cases} x_1 + 5/4\, x_2 - 1/2\, x_3 = 5/2 \\ x_2 = 4 \\ x_3 = 3 \end{cases}$$

which simpler than the original system we started with

But has exactly the same solution, as the original system.

Problem 2: we need to know the following result:

$\det(A) \neq 0 \iff Ax = 0$ has only trivial solution.

This means: If $\det(A) \neq 0$, then the only vector x that satisfies $Ax=0$ is the zero vector, said a bit differently:

If $\det(A) \neq 0$, then, $A\vec{x} = 0 \Rightarrow \vec{x} = \vec{0}$

This also says:

If $\det(A) = 0$, then there exists a NON ZERO vector x, such that $Ax = 0$. In other words

If $\det(A) = 0$, then $Ax = 0$ has a <u>nonzero</u> solution.

a) $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$, $\det(A) = 1(4) - 3(-2) = 10$

$\det A \neq 0 \Rightarrow Ax = 0$ has only trivial solution. $(x=0)$.

In other words $Ax=0$ <u>Does Not</u> have a nonzero solution.

b) $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$ $\det(A) = 3(2) - (-6)(-1) = 0$

$\det(A) = 0 \Rightarrow Ax=0$ has a nonzero solution.

c) $A = \begin{pmatrix} \boxed{1} & \boxed{3} & \boxed{4} \\ -2 & -5 & -3 \\ 1 & 4 & 9 \end{pmatrix}$ $\det(A) = 1\begin{vmatrix} -5 & -3 \\ 4 & 9 \end{vmatrix} - 3\begin{vmatrix} -2 & -3 \\ 1 & 9 \end{vmatrix} + 4\begin{vmatrix} -2 & -5 \\ 1 & 4 \end{vmatrix}$

$= (-5)(9) - (-3)(4)$

$(-3)[(-2)(9) - (-3)(1)]$

$+ 4[(-2)(4) - (-5)(1)]$

$= 0$

$\det(A) = 0 \Rightarrow Ax = 0$ has non zero solution.

d) $A = \begin{pmatrix} \boxed{1 \quad 1 \quad 1} \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$

$$\det(A) = 1\begin{vmatrix} -2 & 2 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= (-2)(-1) - (2)(2) - [(1)(-1) - (2)(1)] + [(1)(2) - (1)(-2)]$$

$$= 5$$

$\det(A) \neq 0 \Rightarrow Ax = 0$ has only the zero solution.

$(Ax = 0 \Rightarrow x = 0)$

Note: A formula for computing the determinant of a 3x3 matrix.

let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \det(A) =$

$$(a)\begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b)\begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c)\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

In this formula by $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ we mean the determinant of a 2x2 submatrix.

so $\begin{vmatrix} e & f \\ h & i \end{vmatrix} = (e)(i) - (f)(h)$ and similarly

$\begin{vmatrix} d & f \\ g & i \end{vmatrix} = (d)(i) - (f)(g)$ and

$\begin{vmatrix} d & e \\ g & h \end{vmatrix} = (d)(h) - (e)(g)$

Problem 3:

$x_1' = 4x_1 - 3x_2$

$x_2' = 2x_1 - 3x_2$

In matrix notation, we have:

$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let $A = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix}$.

First find eigen-values of A:

need to solve $\det(A - \lambda I) = 0$, $A - \lambda I = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

(note $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

$= \begin{pmatrix} 4-\lambda & -3 \\ 2 & -3-\lambda \end{pmatrix}$

so $\det(A - \lambda I) = (4-\lambda)(-3-\lambda) - (-3)(2)$

$= \lambda^2 - \lambda - 6 = 0 \Rightarrow$

$(\lambda - 3)(\lambda + 2) = 0 \Rightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = -2 \end{cases}$

Then for each eigen value, Find a corresponding eigen vector,

For $\lambda_1 = 3$, need to find a <u>non zero</u> vector V_1, such that

$A\vec{V_1} = \lambda_1 \vec{V_1}$ so

$A V_1 = 3 V_1$ or $A V_1 = 3 I V_1$ or $A v_1 - 3 I v_1 = 0$ or

$(A - 3I) V_1 = 0$, so we need to form the matrix $A - 3I$

and solve the equation. note that we already know

$A - \lambda I = \begin{pmatrix} 4-\lambda & -3 \\ 2 & -3-\lambda \end{pmatrix}$, now since $\lambda = 3$, $A - 3I = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

so we need to solve $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} V_1 = 0$

let $V_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and note that 0 in above is the 0 vector: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

so $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ multiply the matrix by the vector to get:

$$\begin{cases} x_1 - 3x_2 = 0 \\ 2x_1 - 6x_2 = 0 \end{cases}$$

note that the second equation is just the first one times 2, so any x_1, x_2 that satisfies the first, also satisfies the second.

so just look at

$x_1 - 3x_2 = 0 \Rightarrow x_1 = 3x_2$.

This is the only constraint on the form of the solution.

so $V_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any number x_2.

we only need one eigen vector, so let $x_2 = 1$ and therefore

$V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Do the same steps to find an eigen vector V_2, corresponding to $\lambda_2 = -2$.

$A \cdot V_2 = \lambda_2 V_2 \Rightarrow (A - \lambda_2 I) V_2 = 0 \Rightarrow (A + 2I) V_2 = 0 \Rightarrow$

$\begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{cases} 6x_1 - 3x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$$

just look at one of the equations. (the 1st one is 3 times the second one)

$\Rightarrow 2x_1 - x_2 = 0 \Rightarrow$

$x_2 = 2x_1 \Rightarrow$

$V_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

let $x_1 = 1 \Rightarrow V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

then the general solution has this form:

$$\vec{X}(t) = c_1 \vec{V_1} e^{\lambda_1 t} + c_2 \vec{V_2} e^{\lambda_2 t}$$

since we have distinct eigen values. so

$$\vec{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$

Problem 4:

$$x' = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} x$$

Find eigen values of the matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - (2)(-1) = 0 \Rightarrow$$

$$\lambda^2 - 4\lambda + 5 = 0 \Rightarrow$$

$$\lambda_{1,2} = (4 \pm \sqrt{16 - 4(1)(5)})/2$$

$\lambda_{1,2} = 2 \pm i$, let $\lambda_1 = 2+i$, $\lambda_2 = 2-i$

Then find an eigen value corresponding to λ_1. In case we have complex eigen values, eigen/vals/vects come in complex conjugate pairs. as you see λ_2 is the complex conjugate of λ_1. also you don't need to find an eigen vector V_2 for λ_2. It's just complex conjugate of the eigen value you find for λ_1. So here we just find V_1 for λ_1.

V_2 is complex conjugate of V_1.

find v_1 for $\lambda_1 = 2+i$:

$$Av_1 = \lambda_1 v_1 \Rightarrow (A - \lambda_1 I)v_1 = 0 \Rightarrow (A - (2-i)I)\,v_1 = 0 \Rightarrow$$

$$\begin{pmatrix} 1-2-i & 2 \\ -1 & 3-2-i \end{pmatrix} v_1 = 0 \Rightarrow \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} v_1 = 0 \Rightarrow \text{let } v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{cases} (-1-i)x_1 + 2x_2 = 0 \\ -x_1 + (1-i)x_2 = 0 \end{cases} \Rightarrow$$

from the second equation: $x_1 = (1-i)x_2$

put this in the first equation:

$$-(1+i)(1-i)x_2 + 2x_2 = 0 \Rightarrow$$

$$-(1^2 - (i)^2)x_2 + 2x_2 = 0 \Rightarrow$$

$$-(1-(-1))x_2 + 2x_2 = 0 \Rightarrow$$

$$-2x_2 + 2x_2 = 0 \Rightarrow$$

$x_2 = x_2$ which is true for all x_2.

so we have $\begin{cases} x_1 = (1-i)x_2 \\ x_2 = x_2 \end{cases}$

$$\Rightarrow v_1 = \begin{pmatrix} (1-i)x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \text{let } x_2 = 1$$

$\Rightarrow v_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \rightarrow$ an eigen value corresponding to λ_1.

then $v_2 = \bar{v}_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \rightarrow$ complex conjugate of v_1.

$$\text{so } \vec{x}(t) = c_1 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(2+i)t} + c_2 \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{(2-i)t}$$

finally to write the solution in terms of real function we look at the formula given in page 344.

choose one of the eigen values and the corresponding eigen vector.

$\lambda_1 = 2+i, \quad V_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

then write $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, let $B_1 = Re(V_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $B_2 = Img\, V_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

let $\alpha = 2$ $(Re\, \lambda_1)$ and $\beta = 1$ $(Img\, \lambda_1)$

Then $x_1 = [B_1 \cos \beta t - B_2 \sin \beta t]\, e^{\alpha t}$ and

$x_2 = [B_2 \cos \beta t + B_1 \sin \beta t]\, e^{\alpha t}$ are linearly independent

solutions of the system of ODEs.

so $\vec{x}_1 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t\right] e^{2t}$

$\vec{x}_2 = \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t\right] e^{2t}$

then the general solution is : $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

so $\vec{x}(t) = c_1 \begin{pmatrix} (\cos t + \sin t)e^{2t} \\ \cos t\, e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} (-\cos t + \sin t)e^{2t} \\ \sin t\, e^{2t} \end{pmatrix}$

note : It does not matter which eigen value/vector you choose, you get the same general solution in either case.

Let's do this computation fo $\lambda_2 = 2 - i$, $V_2 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

although it's not required and we get the same $\vec{x}(t)$:

$\lambda_2 = 2 - i \qquad v_2 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then $\alpha = 2 \quad (\mathrm{Re}\,\lambda_2)$, $\beta = -1 \quad (\mathrm{Img}\,\lambda_2)$

$B_1 = \mathrm{Re}(v_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\quad B_2 = \mathrm{Img}(v_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then :

$$\vec{x}_1(t) = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(-t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(-t) \right] e^{2t}$$

$$\vec{x}_2(t) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(-t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(-t) \right] e^{2t}$$

<u>note</u> : $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$

so
$$\vec{x}_1(t) = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] e^{2t}$$

$$\vec{x}_2(t) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t \right] e^{2t}$$

then $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

so $$\vec{x}(t) = c_1 \begin{pmatrix} (\cos t + \sin t)\, e^{2t} \\ \cos t \; e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} (\cos t - \sin t)\, e^{2t} \\ -\sin t \; e^{2t} \end{pmatrix}$$

So $$\vec{x}(t) = c_1 \begin{pmatrix} (\cos t + \sin t)\, e^{2t} \\ \cos t \; e^{2t} \end{pmatrix} - c_2 \begin{pmatrix} (-\cos t + \sin t)\, e^{2t} \\ \sin t \; e^{2t} \end{pmatrix}$$

but $-c_2$ is arbitrary constant, let $-c_2 = c_3$

and compare this solution to the previous and observe

that they are the same.

Problem 5:

$$x' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} x$$

Find eigen values of $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$.

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(7-\lambda) - (3)(-3) = 0 \Rightarrow$

$\lambda^2 - 8\lambda + 16 = 0 \Rightarrow (\lambda - 4)(\lambda - 4) = 0 \Rightarrow \lambda_1 = \lambda_2 = 4$

So we get a <u>repeated</u> eigen value.

look for an eigen vector $V_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$AV_1 = \lambda_1 V_1 \Rightarrow (A - 4I)V_1 = 0 \Rightarrow \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$\begin{cases} -3x_1 - 3x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{cases}$ look at one of the equations:

$3x_1 + 3x_2 = 0 \Rightarrow x_1 = -x_2 \Rightarrow V_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Although for different choices of x_1, we find different eigen vectors, But, they are just constant multiples of each other, i.e, They are <u>not</u> <u>linearly Independent</u>.

Now, we seek another vector $\vec{P}$ such that:

$$(A - \lambda_1 I)\vec{P} = \vec{V_1}$$

lets choose $x_1 = 1 \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So $(A-4I)\vec{P} = \begin{pmatrix}1\\-1\end{pmatrix}$, let $\vec{P} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$, then

$$\begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow$$

$$\begin{cases} -3x_1 - 3x_2 = 1 \\ 3x_1 + 3x_2 = -1 \end{cases}$$

the second equation is just the first one times -1, so just look at one of them:

$$3x_1 + 3x_2 = -1 \Rightarrow 3x_1 = -1 - 3x_2 \Rightarrow x_1 = \frac{-1-3x_2}{3}$$

so $\vec{P} = \begin{pmatrix}\frac{-1-3x_2}{3}\\ x_2\end{pmatrix} = x_2\begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}-1/3\\0\end{pmatrix}$ ⊛

let $x_2 = 0 \Rightarrow \vec{P} = \begin{pmatrix}-1/3\\0\end{pmatrix}$

note that ⊛ step was not required but I don't have white out!

once you have $\vec{P} = \begin{pmatrix}\frac{-1-3x_2}{3}\\ x_2\end{pmatrix}$ just let $x_2 = 0 \Rightarrow$

$\vec{P} = \begin{pmatrix}-1/3\\0\end{pmatrix}$ (you can choose any other value for x_2)

Now, the general solution In this case has the form:

$$\vec{X}(t) = c_1\vec{V}_1 e^{\lambda_1 t} + c_2\left[t\vec{V}_1 e^{\lambda_1 t} + \vec{P}e^{\lambda_1 t}\right] \quad \text{so:}$$

$$\vec{X}(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{4t} + c_2\left[t\begin{pmatrix}1\\-1\end{pmatrix}e^{4t} + \begin{pmatrix}-1/3\\0\end{pmatrix}e^{4t}\right].$$

Problem 6 :

$$x' = \begin{pmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix} x$$

find eigen-values of $A = \begin{pmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 4 & 4 \\ -1 & -2-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$-1 \begin{vmatrix} 4 & 4 \\ -2-\lambda & 0 \end{vmatrix} - 0 \begin{vmatrix} 2-\lambda & 4 \\ -1 & 0 \end{vmatrix} + (-2-\lambda) \begin{vmatrix} 2-\lambda & 4 \\ -1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$-1\,[4(0) - 4(-2-\lambda)] + (-2-\lambda)\,[(2-\lambda)(-2-\lambda) - (-4)] = 0 \Rightarrow$$

$$-4(\lambda+2) - (\lambda+2)\,[-(2-\lambda)(2+\lambda) + 4] = 0 \Rightarrow$$

$$-4(\lambda+2) + (\lambda+2)\,[(2-\lambda)(2+\lambda) - 4] = 0 \Rightarrow$$

$$(\lambda+2)\,[\cancel{-4} + \cancel{2^2} - \lambda^2 - 4] = 0 \Rightarrow -(\lambda+2)(\lambda^2+4) = 0 \Rightarrow$$

$$\lambda_1 = -2, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i$$

Find an eigen value corresponding to $\lambda_1 = -2$.

$$(A - \lambda_1 I)\,V_1 = 0 \Rightarrow (A + 2I)\,V_1 = 0 \Rightarrow \begin{pmatrix} 4 & 4 & 4 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} V_1 = 0$$

let $V_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ then obviously from the second and the third row, we find:

$$-x_1 = 0 \Rightarrow x_1 = 0$$

from the first row :

$$\underbrace{\cancel{4x_1}}_{0} + 4x_2 + 4x_3 = 0 \Rightarrow 4x_2 + 4x_3 = 0 \Rightarrow x_2 = -x_3$$

So $x_1 = 0$, $x_2 = -x_3$. then $\vec{v}_1 = \begin{pmatrix} 0 \\ x_2 \\ -x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

letting $x_2 = 1$, we get $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Now, we find an eigen-vector corresponding to $\lambda_2 = 2i$.

$(A - \lambda_2 I)\vec{v}_2 = 0 \Rightarrow$ let $v_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, we have:

$$\begin{pmatrix} 2-2i & 4 & 4 \\ -1 & -2-2i & 0 \\ -1 & 0 & -2-2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from the second and the last row, we see that

$-x_1 + (-2-2i)x_2 = 0$ & $-x_1 + (-2-2i)x_3 = 0 \Rightarrow$ comparing these

two equations shows that $x_2 = x_3$.

so we have from the first row:

$(2-2i)x_1 + 4\cancel{x_2} + 4x_3 = 0 \Rightarrow (2-2i)x_1 + 8x_2 = 0 \Rightarrow x_1 = \dfrac{8x_2}{2i-2} = \dfrac{4x_2}{i-1}$

note that $\dfrac{4x_2}{i-1} = \dfrac{4x_2}{i-1} \cdot \dfrac{i+1}{i+1} = \dfrac{4x_2(i+1)}{i^2-1^2} = \dfrac{4x_2(i+1)}{-2} = -2x_2(i+1)$

Thus $x_1 = -2x_2(i+1)$.

so $\vec{v}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2(i+1) \\ x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2i-2 \\ 1 \\ 1 \end{pmatrix}$

now let $x_2 = 1 \Rightarrow \vec{v}_2 = \begin{pmatrix} -2i-2 \\ 1 \\ 1 \end{pmatrix}$

We need no extra work to find $\vec{v}_3$; the eigen value corresponding to $\lambda_3 = -2i$. It's just the complex conjugate of $\vec{v}_2$. So $\vec{v}_3 = \begin{pmatrix} 2i-2 \\ 1 \\ 1 \end{pmatrix}$.

Putting everything together, we have:

$$\lambda_1 = -2, \ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad ; \quad \lambda_2 = 2i, \ \vec{v}_2 = \begin{pmatrix} -2i-2 \\ 1 \\ 1 \end{pmatrix} \quad ; \quad \lambda_3 = -2i, \ \vec{v}_3 = \begin{pmatrix} 2i-2 \\ 1 \\ 1 \end{pmatrix}$$

Thus the general solution looks like:

$$\vec{X}(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2i-2 \\ 1 \\ 1 \end{pmatrix} e^{2it} + c_3 \begin{pmatrix} 2i-2 \\ 1 \\ 1 \end{pmatrix} e^{-2it}. \quad *$$

we can write the general solution in terms of Real functions:

look at $\lambda_2 = 2i, \ \vec{V}_2 = \begin{pmatrix} -2i-2 \\ 1 \\ 1 \end{pmatrix}$ (you could choose $\lambda_3, \vec{v}_3$).

then $\lambda_2 = 0 + 2i \Rightarrow \alpha = 0, \ \beta = 2$.

Also $Re(\vec{v}_2) = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = B_1$ and $Img(\vec{v}_2) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$

((note: $\vec{v}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$))

Then By the formulas we saw before, we define:

$$\vec{X}_1(t) = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \sin 2t \right] e^{(0)t}$$

$$\vec{X}_2(t) = \left[\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \sin 2t \right] e^{(0)t}$$

then the combination * is the same as $c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$,

that is: $c_1 \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}$

Finally the general solution in Real form is:

$$\vec{X}(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + c_3 \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$