Math 527 - Homework 7 Solutions
4/16/14

Problem 1. Find the power series expansions of $\sin x$ and $\cos x$ about $x = 0$ by using the Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Solution. Let $f(x) = \sin x$. We calculate

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Plugging these into the formula for the Taylor expansion gives

$$\begin{aligned} f(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

Let $g(x) = \cos x$. We calculate

$$\begin{aligned} g(x) &= \cos x & g(0) &= 1 \\ g'(x) &= -\sin x & g'(0) &= 0 \\ g''(x) &= -\cos x & g''(0) &= -1 \\ g^{(3)}(x) &= \sin x & g^{(3)}(0) &= 0 \\ g^{(4)}(x) &= \cos x & g^{(4)}(0) &= 1 \end{aligned}$$

Plugging these into the formula for the Taylor expansion gives

$$\begin{aligned} g(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. \end{aligned}$$

Problem 2. Use the power series expansions of $\sin x$ and $\cos x$ to show that $\frac{d}{dx}\sin x = \cos x$.

Solution.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\
&= \cos x.
\end{aligned}$$

Problem 3. Find the general solution of the ODE using the ansatz $y = e^{\lambda x}$ and then find it again using the power series method:

$$y'' + k^2 y = 0.$$

Solution. We first solve by using the ansatz $y = e^{\lambda x}$. This gives the quadratic equation $\lambda^2 + k^2 = 0$, which implies $\lambda = \pm ki$. Therefore, the general solution of the ODE is $y = c_0 \cos(kx) + c_1 \sin(kx)$.

Now we use the ansatz $y = \sum_{n=0}^{\infty} c_n x^n$, which implies $y = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Plugging into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} k^2 c_n x^n = 0.$$

Manipulating the left-hand side gives

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} k^2 c_n x^n &= 0 \\ \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} + k^2 c_n \right] x^n &= 0. \end{aligned}$$

We conclude from this equation that

$$(n+2)(n+1) c_{n+2} + k^2 c_n = 0$$

for all $n \geq 0$. Solving for c_{n+2} gives the recurrence relation

$$c_{n+2} = -\frac{k^2 c_n}{(n+2)(n+1)}.$$

Given c_0, c_1, we calculate

$$\begin{aligned} &n = 0\text{:} \quad c_2 = -\tfrac{k^2}{2!} c_0 && n = 1\text{:} \quad c_3 = -\tfrac{k^2}{3!} c_1 \\ &n = 2\text{:} \quad c_4 = \tfrac{k^4}{4!} c_0 && n = 3\text{:} \quad c_5 = \tfrac{k^4}{5!} c_1 \\ &n = 4\text{:} \quad c_6 = -\tfrac{k^6}{6!} c_0 && n = 5\text{:} \quad c_7 = \tfrac{k^6}{7!} c_1. \end{aligned}$$

Therefore, the solution to the ODE is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 \left(1 - \frac{k^2}{2!}x^2 + \frac{k^4}{4!}x^4 - \frac{k^6}{6!}x^6 + \cdots\right) \\
&\qquad + c_1 \left(x - \frac{k^2}{3!}x^3 + \frac{k^4}{5!}x^5 - \frac{k^6}{7!}x^7 + \cdots\right) \\
&= c_0 \cos(kx) + \frac{c_1}{k}\left(kx - \frac{k^3}{3!}x^3 + \frac{k^5}{5!}x^5 - \frac{k^7}{7!}x^7 + \cdots\right) \\
&= c_0 \cos(kx) + \hat{c}_1 \sin(kx).
\end{aligned}$$

Problem 4. Find two linearly independent power series solutions of the ODE centered about $x = 0$:

$$y'' + x^2 y' + xy = 0.$$

If the power series does not simplify to a known function or have a simple expression for the coefficients, provide the first four terms of each solution.

Solution. We use the ansatz $y = \sum_{n=0}^{\infty} c_n x^n$, which implies $y = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Plugging into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n = 0.$$

Manipulating the left-hand side gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) c_{n-1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n &= 0 \\ 2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} (n-1) c_{n-1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n &= 0 \\ 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} + n c_{n-1} \right] x^n &= 0. \end{aligned}$$

We conclude from this equation that $2c_2 = 0$, hence $c_2 = 0$, and

$$(n+2)(n+1) c_{n+2} + n c_{n-1} = 0$$

for all integers $n \geq 1$. Solving for c_{n+2} gives the recurrence relation

$$c_{n+2} = -\frac{n c_{n-1}}{(n+2)(n+1)}.$$

Given c_0, c_1 we have

$$
\begin{aligned}
& & c_2 &= 0 \\
n = 1: & & c_3 &= -\frac{1}{3\cdot 2}c_0 \\
n = 2: & & c_4 &= -\frac{2}{4\cdot 3}c_1 \\
n = 3: & & c_5 &= 0 \\
n = 4: & & c_6 &= \frac{4}{6\cdot 5\cdot 3\cdot 2}c_0 \\
n = 5: & & c_7 &= \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}c_1 \\
n = 6: & & c_8 &= 0 \\
n = 7: & & c_9 &= -\frac{7\cdot 4}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}c_0 \\
n = 8: & & c_{10} &= -\frac{8\cdot 5\cdot 2}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}c_1.
\end{aligned}
$$

Therefore, our solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0\left(1 - \frac{1}{3\cdot 2}x^3 + \frac{4}{6\cdot 5\cdot 3\cdot 2}x^6 - \frac{7\cdot 4}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \cdots\right) \\
&\quad + c_1\left(x - \frac{2}{4\cdot 3}x^4 + \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}x^7 - \frac{8\cdot 5\cdot 2}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots\right).
\end{aligned}
$$

Two linearly independent solutions to the ODE are

$$y_0 = 1 - \frac{1}{3\cdot 2}x^3 + \frac{4}{6\cdot 5\cdot 3\cdot 2}x^6 - \frac{7\cdot 4}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \cdots$$

and

$$y_1 = x - \frac{2}{4\cdot 3}x^4 + \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}x^7 - \frac{8\cdot 5\cdot 2}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots .$$

Problem 5. Use the power series method to solve the initial value problem and specify the solution's interval of convergence.

$$(x-1)y'' - xy' + y = 0, \qquad y(0) = -2,\ \ y'(0) = 6.$$

Solution. We use the ansatz $y = \sum_{n=0}^{\infty} c_n x^n$, which implies $y = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Plugging into the ODE gives

$$(x-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Manipulating the left-hand side gives

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n &= 0\\
\sum_{n=1}^{\infty}(n+1)n c_{n+1} x^n - \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n &= 0\\
\sum_{n=0}^{\infty}(n+1)n c_{n+1} x^n - \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n &= 0\\
\sum_{n=0}^{\infty}\left[(n+1)n c_{n+1} - (n+2)(n+1)c_{n+2} - c_n(n-1)\right]x^n &= 0.
\end{aligned}$$

We conclude from this equation that

$$(n+1)n c_{n+1} - (n+2)(n+1)c_{n+2} - (n-1)c_n = 0$$

for all integers $n \geq 0$. Solving for c_{n+2} gives the recurrence relation

$$c_{n+2} = \frac{(n+1)n c_{n+1} - (n-1)c_n}{(n+2)(n+1)}.$$

The initial values $y(0) = -2$ and $y'(0) = 6$ tell us that $c_0 = -2$ and $c_1 = 6$.

We can now use our recurrence relation to compute higher-order coefficients.

$$\begin{aligned}
n = 0: &\qquad c_2 = -1 \\
n = 1: &\qquad c_3 = -\frac{1}{3} \\
n = 2: &\qquad c_4 = -\frac{1}{4 \cdot 3} \\
n = 3: &\qquad c_5 = -\frac{1}{5 \cdot 4 \cdot 3} \\
n = 4: &\qquad c_6 = -\frac{1}{6 \cdot 5 \cdot 4 \cdot 3}.
\end{aligned}$$

We see a pattern emerge; $c_k = -\frac{2}{k!}$ for $k \geq 2$. Therefore, our solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n \\
&= -2 + 6x + \sum_{n=2}^{\infty} \left(-\frac{2}{n!} \right) x^n \\
&= -2 + 6x - 2 \sum_{n=2}^{\infty} \frac{x^n}{n!} \\
&= -2 + 6x - 2 \left[\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) - 1 - x \right] \\
&= -2 + 6x - 2(e^x - 1 - x) \\
&= 8x - 2e^x.
\end{aligned}$$

This solution clearly converges for all x.