Problem 1 (a) 5 pts $\mathcal{L}\{(3t+1)\mathcal{U}(t-1)\}$, using $\mathcal{L}\{\mathcal{U}(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t)|_{t\to t+a}\}$

$= e^{-s}\mathcal{L}\{(3t+1)|_{t\to t+1}\}$ $\quad a=1$

$= e^{-s}\mathcal{L}\{3(t+1)+1\} = e^{-s}\mathcal{L}\{3t+4\} = e^{-s}[3\mathcal{L}\{t\}+4\mathcal{L}\{1\}]$

$= e^{-s}[\frac{3}{s^2}+\frac{4}{s}]$

5 pts

(b) $\mathcal{L}\{e^{2t}(t-1)^2\}$, using $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s\to s-a}$

$= \mathcal{L}\{(t-1)^2\}|_{s\to s-a}$, $a=2$

$= \mathcal{L}\{t^2-2t+1\}|_{s\to s-2}$

$= \mathcal{L}\{t^2\}|_{s\to s-2} - 2\mathcal{L}\{t\}|_{s\to s-2} + \mathcal{L}\{1\}|_{s\to s-2}$

$= \frac{2}{s^3}|_{s\to s-2} - 2\cdot\frac{1}{s^2}|_{s\to s-2} + \frac{1}{s}|_{s\to s-2}$

$= \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}$

5 pts

(c) $\mathcal{L}^{-1}\{\frac{2s+5}{s^2+6s+34}\} = \mathcal{L}^{-1}\{\frac{2(s+3)-1}{(s+3)^2+25}\}$, using $\mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}\cdot\mathcal{L}^{-1}\{F(s)\}$

$= \mathcal{L}^{-1}\{\frac{2s-1}{s^2+25}|_{s\to s+3}\} = e^{-3t}\cdot\mathcal{L}^{-1}\{\frac{2s-1}{s^2+25}\}$

$= e^{-3t}\cdot[2\mathcal{L}^{-1}\{\frac{s}{s^2+25}\} - \mathcal{L}^{-1}\{\frac{1}{s^2+25}\}]$

$= e^{-3t}\cdot[2\cos 5t - \frac{1}{5}\sin 5t]$

5 pts

(d) $\mathcal{L}^{-1}\{\frac{se^{-\pi s/2}}{s^2+4}\}$, using $\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{U}(t-a)[\mathcal{L}^{-1}\{F(s)\}]_{t\to t-a}$

$= \mathcal{U}(t-\frac{\pi}{2})\cdot[\mathcal{L}^{-1}\{\frac{s}{s^2+4}\}]_{t\to t-\frac{\pi}{2}}$

$= \mathcal{U}(t-\frac{\pi}{2})\cdot[\cos 2t]_{t\to t-\frac{\pi}{2}} = \mathcal{U}(t-\frac{\pi}{2})\cdot\cos(2t-\pi) = -\mathcal{U}(t-\frac{\pi}{2})\cos 2t$

Problem 2. (a) 10 pts $f(t) = \begin{cases} \sin t & 0\le t<2\pi \\ 0 & 2\pi\le t\end{cases}$, rewrite $f(t) = \sin t\cdot[1-\mathcal{U}(t-2\pi)]$

$= \sin t - \sin t\cdot\mathcal{U}(t-2\pi)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t\cdot\mathcal{U}(t-2\pi)\}$

$= \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin t|_{t\to t+2\pi}\}$

$= \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$, $\sin(t+2\pi) = \sin t$

$= \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$

2. (b) 10 pts $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & 1 \le t \end{cases}$, rewrite $f(t) = u(t-1)\cdot t^2$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-1)\cdot t^2\}$

$= e^{-s}\mathcal{L}\{t^2|_{t\to t+1}\} = e^{-s}\mathcal{L}\{t^2+2t+1\} = e^{-s}\cdot\mathcal{L}\{t^2\} + 2e^{-s}\cdot\mathcal{L}\{t\} + e^{-s}\cdot\mathcal{L}\{1\}$

$= \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s}$

Problem 5. 20 pts $y' + 2y = f(t)$, $y(0) = 0$, where $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t \end{cases}$

$\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{f(t)\}$

$sY(s) - y(0) + 2Y(s) = \mathcal{L}\{f(t)\}$

$(s+2)Y(s) = \mathcal{L}\{f(t)\}$

$\Rightarrow Y(s) = \frac{1}{(s+2)s^2} - \frac{e^{-s}}{s+2}\left[\frac{1}{s^2} + \frac{1}{s}\right]$

$\begin{cases} f(t) = t\cdot[1-u(t-1)] = t - t\cdot u(t-1) \\ \mathcal{L}\{f(t)\} = \frac{1}{s^2} - \mathcal{L}\{u(t-1)\cdot t\} \\ \quad = \frac{1}{s^2} - e^{-s}\cdot\mathcal{L}\{t|_{t\to t+1}\} \\ \quad = \frac{1}{s^2} - e^{-s}\cdot\left[\frac{1}{s^2} + \frac{1}{s}\right] \end{cases}$

$\frac{1}{(s+2)s^2} = \frac{A}{s+2} + \frac{Bs+C}{s^2}$

$= \frac{As^2 + Bs^2 + 2Bs + Cs + 2C}{(s+2)s^2}$ $\therefore$ $A+B=0$, $2B+C=0$, $2C=1$

then $C = \frac{1}{2}$, $B = -\frac{1}{4}$, $A = \frac{1}{4}$

$\frac{1}{(s+2)s^2} = \frac{\frac{1}{4}}{s+2} + \frac{-\frac{1}{4}s + \frac{1}{2}}{s^2}$, $\frac{1}{s(s+2)} = \frac{1}{2}\cdot\left[\frac{1}{s} - \frac{1}{s+2}\right]$

(*) $Y(s) = \frac{\frac{1}{4}}{s+2} + \frac{-\frac{1}{4}s+\frac{1}{2}}{s^2} - e^{-s}\left[\frac{\frac{1}{4}}{s+2} + \frac{-\frac{1}{4}s+\frac{1}{2}}{s^2} + \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right]$ + 10 pts

① $\mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s+2}\right\} = \frac{1}{4}e^{-2t}$; ② $\mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}s+\frac{1}{2}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2}\right\} = -\frac{1}{4} + \frac{1}{2}t$;

③ $\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{\frac{1}{4}}{s+2}\right\} = u(t-1)\cdot\left[\mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s+2}\right\}\right]_{t\to t-1}$

$= u(t-1)\cdot\left[\frac{1}{4}e^{-2t}\right]_{t\to t-1} = u(t-1)\cdot\frac{1}{4}e^{-2(t-1)}$

④ $\mathcal{L}^{-1}\left\{e^{-s}\cdot\left[\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2}\right]\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{e^{-s}}{s} + \frac{1}{2}\frac{e^{-s}}{s^2}\right\}$

$= -\frac{1}{4}u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right]_{t\to t-1} + \frac{1}{2}u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right]_{t\to t-1}$

$= -\frac{1}{4}u(t-1)\cdot 1 + \frac{1}{2}u(t-1)\cdot(t-1)$

⑤ $\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{2}\frac{1}{s}\right\} = \frac{1}{2}u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right]_{t\to t-1} = \frac{1}{2}u(t-1)$

⑥ $\mathcal{L}^{-1}\left\{e^{-s}\cdot\left(-\frac{1}{2}\right)\frac{1}{s+2}\right\} = -\frac{1}{2}u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\right]_{t\to t-1} = -\frac{1}{2}u(t-1)e^{-2(t-1)}$

Combine all the above six terms together, + 10 pts

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t - u(t-1)\frac{1}{4}e^{-2(t-1)} + \frac{1}{4}u(t-1) - \frac{1}{2}u(t-1)\cdot(t-1) - \frac{1}{2}u(t-1)$ $+ \frac{1}{2}u(t-1)e^{-2(t-1)}$

Simplify $y(t) = \frac{1}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t - \frac{1}{4}u(t-1) - \frac{1}{2}u(t-1)(t-1) + \frac{1}{4}u(t-1)e^{-2(t-1)}$

$y(t) = \begin{cases} \frac{1}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t, & 0 \le t < 1 \\ \frac{1}{4}e^{-2t} + \frac{1}{4}e^{-2(t-1)}, & t \geq 1 \end{cases}$

20pts

6. $y'' + 2y' + y = f(t)$, $y(0)=0$, $y'(0)=1$, where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & 3 \le t \end{cases}$

$\Rightarrow \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$

$s^2 Y(s) - s y(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \mathcal{L}\{f(t)\}$, $f(t) = 2 \cdot u(t-3)$

$[s^2+2s+1]Y(s) = 1 + \frac{2e^{-3s}}{s}$ $\mathcal{L}\{f(t)\} = \frac{2e^{-3s}}{s}$

$Y(s) = \frac{1}{s^2+2s+1} + 2e^{-3s} \cdot \frac{1}{s(s+1)^2}$

$\frac{1}{s(s+1)^2} = \frac{1}{s+1} \cdot [\frac{1}{s} - \frac{1}{s+1}] = \frac{1}{s(s+1)} - \frac{1}{(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$

$\therefore Y(s) = \frac{1}{(s+1)^2} + 2e^{-3s} \cdot [\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}]$

$y(t) = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} + 2\mathcal{L}^{-1}\{e^{-3s} \cdot \frac{1}{s}\} - 2\mathcal{L}^{-1}\{e^{-3s} \cdot \frac{1}{s+1}\} - 2\mathcal{L}^{-1}\{e^{-3s} \cdot \frac{1}{(s+1)^2}\}$

$= e^{-t} \cdot t + 2u(t-3) \cdot 1 - 2u(t-3) \cdot e^{-(t-3)} - 2u(t-3) \cdot [\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}]_{t \to t-3}$

$\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} = \mathcal{L}^{-1}\{\frac{1}{s^2}|_{s \to s+1}\} = e^{-t}\mathcal{L}^{-1}\{\frac{1}{s^2}\} = e^{-t} \cdot t$

$[\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}]_{t \to t-3} = e^{-(t-3)}(t-3)$

$y(t) = e^{-t} \cdot t + 2u(t-3) - 2u(t-3)e^{-(t-3)} - 2u(t-3)[e^{-(t-3)}(t-3)]$

$y(t) = \begin{cases} e^{-t} \cdot t, & 0 \le t < 3 \\ e^{-t} \cdot t + 2 + 4e^{-(t-3)} - 2e^{-(t-3)} \cdot t, & 3 \le t \end{cases}$

20pts

7. $y'' + 4y' + 5y = \delta(t - 2\pi)$, $y(0) = y'(0) = 0$

$\Rightarrow \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2\pi)\}$

$s^2 Y(s) - s y(0) - y'(0) + 4[sY(s) - y(0)] + 5Y(s) = e^{-s \cdot 2\pi}$

$[s^2 + 4s + 5]Y(s) = e^{-s \cdot 2\pi}$

$Y(s) = e^{-2\pi s} \cdot \frac{1}{s^2+4s+5}$

(+10pts) $= e^{-2\pi s} \cdot \frac{1}{(s+2)^2+1}$

$y(t) = \mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{(s+2)^2+1}\}$

$= u(t-2\pi) \cdot [\mathcal{L}^{-1}\{\frac{1}{(s+2)^2+1}\}]_{t \to t-2\pi}$

$\because \mathcal{L}^{-1}\{\frac{1}{(s+2)^2+1}\} = \mathcal{L}^{-1}\{\frac{1}{s^2+1}|_{s \to s+2}\} = e^{-2t} \cdot \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = e^{-2t} \cdot \sin t$

$\therefore y(t) = u(t-2\pi) \cdot [e^{-2t} \cdot \sin t]_{t \to t-2\pi}$

$= u(t-2\pi) \cdot [e^{-2(t-2\pi)} \cdot \sin(t-2\pi)]$

(+10pts) $= u(t-2\pi) \cdot [e^{-2(t-2\pi)} \sin t]$

$y(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ e^{-2(t-2\pi)} \sin t, & 2\pi \le t \end{cases}$