Math 527, HW5, Laplace Transforms.

Problem 1

(a) $\mathcal{L}\{4t^2 - 5\sin 3t\} = \mathcal{L}\{4t^2\} + \mathcal{L}\{-5\sin 3t\}$

$= 4\mathcal{L}\{t^2\} - 5\mathcal{L}\{\sin 3t\} = 4 \cdot \frac{2!}{s^3} - 5 \cdot \frac{3}{s^2+9}$

(b) $\mathcal{L}\{(1+e^{5t})^2\} = \mathcal{L}\{1 + e^{10t} + 2e^{5t}\} =$

$\mathcal{L}\{1\} + \mathcal{L}\{e^{10t}\} + 2\mathcal{L}\{e^{5t}\} = \frac{1}{s} + \frac{1}{s-10} + 2 \cdot \frac{1}{s-5}$

(c) $\mathcal{L}\{t^2 e^{-3t}\} = \left[\mathcal{L}\{t^2\}\right]_{s \to s+3} = \left[\frac{2!}{s^3}\right]_{s \to s+3} = \frac{2}{(s+3)^2}$

(d) $\mathcal{L}\{\sin(3t+2)\} =$, Note: $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\mathcal{L}\{\sin(\underbrace{3t}_{a} + \underbrace{2}_{b})\} = \mathcal{L}\{\sin 3t \cdot \cos 2 + \sin 2 \cdot \cos 3t\}$

$= \cos 2\, \mathcal{L}\{\sin 3t\} + \sin 2\, \mathcal{L}\{\cos 3t\}$

Note: Since $\sin 2$, $\cos 2$ are constants, the above is valid.

$= \cos 2 \cdot \frac{3}{s^2+9} + \sin 2 \cdot \frac{s}{s^2+9}$

Problem 2

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathcal{L}^{-1}\left\{\frac{6!}{s^7} \cdot \frac{1}{6!}\right\} = \frac{1}{6!}\mathcal{L}^{-1}\left\{\frac{6!}{s^7}\right\}$

$= \frac{1}{6!} t^6$

(b) $\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+1+2s}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^3} + \frac{2}{s^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{2!}{s^3} \cdot \frac{1}{2!}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 1 + \frac{1}{2!} t^2 + 2t.$

(c) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{s^2+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t.$

Note: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}$

$= \frac{1}{\sqrt{2}}\sin\sqrt{2}t.$

(d) $\mathcal{L}^{-1}\left\{\frac{10s}{s^2-16}\right\} =$ We first use partial Fraction decomposition.

$$\frac{10s}{s^2-16} = \frac{10s}{(s-4)(s+4)} = \frac{A}{s-4} + \frac{B}{s+4}$$

multiply both sides by $(s-4)$ and put $s=4$. (In Your Head!)

$$A = \frac{10 \cdot 4}{4+4} = \frac{40}{8} = 5$$

then multiply both sides by $(s+4)$ and put $s=-4$.

$$B = \frac{10 \cdot (-4)}{-4-4} = \frac{-40}{-8} = 5$$

so $\mathcal{L}^{-1}\left\{\frac{10s}{s^2-16}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s-4} + \frac{5}{s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s-4}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{s+4}\right\}$

$= 5\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = 5e^{4t} + 5e^{-4t}.$

(e) $\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\} =$ Again use partial fractions.

$$\underbrace{\frac{2s-4}{(s^2+s)(s^2+1)}}_{} = \underbrace{\frac{2s-4}{s(s+1)(s^2+1)}}_{} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

Note: The denominator must be FULLY Factorized.

To find A, multiply both sides by s and put $s=0$.

$$A = \frac{2(0)-4}{(0+1)(0^2+1)} = \frac{-4}{1} = -4$$

To find B, multiply both sides by $s+1$ and put $s=-1$.

$$B = \frac{2(-1)-4}{(-1)((-1)^2+1)} = \frac{-2-4}{-2} = 3$$

To find C, multiply by s (both sides) and then take the limit of both sides as $s \to \infty$, you get:

$$0 = A + B + C \Rightarrow C = -A - B = 4 - 3 = 1$$

To find D, pick a convinient number, say $s=1$.

$$\frac{2(1)-4}{(1)(1+1)((1)^2+1)} = \frac{-4}{1} + \frac{3}{2} + \frac{1(1)+D}{2} \Rightarrow$$

$$\frac{-2}{4} = -2 + D \Rightarrow D = 2 - \frac{2}{4} = \frac{3}{2}$$

so: $\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-4}{s} + \frac{3}{s+1} + \frac{s+3/2}{s^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{-\frac{4}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} + \frac{3/2}{s^2+1}\right\} =$$

$$-4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -4 + 3e^{-t} + \cos t + \frac{3}{2}\sin t.$$

(F) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$ = Note that the denominator is not reducable to Real linear factors, so I'll use completing the square trick.

$$\frac{s}{s^2+2s+5} = \frac{s}{(s^2+2s+4)+1} = \frac{s}{(s+2)^2+1}$$

so we need to find $\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} =$

$$\mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+1}\Big|_{s\to s+2}\right\} =$$

$(s\to s-(-2))$

$$e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}-\frac{2}{s^2+1}\right\} =$$

$$e^{-2t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right) =$$

$$e^{-2t}(\cos t - 2\sin t).$$

<u>Note</u> $\mathcal{L}^{-1}\left\{F(s)\Big|_{s\to s-a}\right\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$.

<u>Problem 3</u>

$$\mathcal{L}\{\sinh(kx)\} = \mathcal{L}\left\{\frac{e^{kx}-e^{-kx}}{2}\right\} =$$

$$\mathcal{L}\left\{\frac{e^{kx}}{2}\right\} + \mathcal{L}\left\{\frac{-e^{-kx}}{2}\right\} = \frac{1}{2}\cdot\frac{1}{s-k} - \frac{1}{2}\cdot\frac{1}{s+k}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{1}{2}\left(\frac{s+k-s+k}{(s-k)(s+k)}\right)$$

$$= \frac{k}{(s-k)(s+k)}.$$

$$\mathcal{L}\{\cosh(kx)\} = \mathcal{L}\left\{\frac{e^{kx}+e^{-kx}}{2}\right\} =$$

$$\frac{1}{2}\left(\mathcal{L}\{e^{kx}\} + \mathcal{L}\{e^{-kx}\}\right) = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$$

$$= \frac{1}{2}\left(\frac{s+k+s-k}{(s-k)(s+k)}\right) = \frac{1}{2}\left(\frac{2s}{(s-k)(s+k)}\right) = \frac{s}{(s-k)(s+k)}.$$

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problem 4

note: $\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty =$

$$\lim_{t\to\infty} -\frac{1}{s}e^{-st} - \left(-\frac{1}{s}e^{-s(0)}\right) = \lim_{t\to\infty} -\frac{1}{s}\cdot\frac{1}{e^{st}} + \frac{1}{s}$$

If $s>0$ then $\frac{1}{e^{st}} \to 0$ as $t\to\infty$

So for $s>0$ we have $\mathcal{L}(1) = \frac{1}{s}$.

Note: $\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt =$ let $u = t^n \Rightarrow du = nt^{n-1}\,dt$

let $dv = e^{-st} \Rightarrow v = -\frac{1}{s}e^{-st}$

using integration by parts we get

$$\int_0^\infty t^n e^{-st}\,dt = t^n\cdot\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)\cdot n\cdot t^{n-1}\,dt$$

we first note that $t^n\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty = \lim_{t\to\infty} -\frac{1}{s}\frac{t^n}{e^{st}} - \frac{1}{s}\frac{(0)^n}{e^{s(0)}}$

If $n>0$ then $-\frac{1}{s}\frac{(0)^n}{e^{s(0)}} = 0$

If $s>0$ $\lim_{t\to\infty} -\frac{1}{s}\frac{t^n}{e^{st}} = 0$ since $t^n \ll e^{st}$

So $t^n\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty = 0$ and

$$\int_0^\infty t^n e^{-st}\,dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

so $\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$. Then

$\mathcal{L}\{t^1\} = \frac{1}{s}\mathcal{L}\{t^{1-1}\} = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$

$\mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t^{2-1}\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$

$\mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{3\cdot 2}{s^4} = \frac{3!}{s^4}$

Guess: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Problem 5

$y' - y = 2\cos 5t$, $y(0) = 0$

$\mathcal{L}\{y' - y\} = \mathcal{L}\{2\cos 5t\} \Rightarrow \mathcal{L}\{y'\} - \mathcal{L}\{y\} = 2\cdot\frac{s}{s^2+25}$

$\Rightarrow s\mathcal{L}\{y\} - \cancel{y(0)}^{\,0} - \mathcal{L}\{y\} = \frac{2s}{s^2+25}$

$\Rightarrow \mathcal{L}\{y\}(s-1) = \frac{2s}{s^2+25}$

$\Rightarrow \mathcal{L}\{y\} = \frac{2s}{(s-1)(s^2+25)}$

$\Rightarrow y = \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+25)}\right\}$ using partial fractions:

$$\frac{2s}{(s-1)(s^2+25)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+25}$$

$$A = \frac{2(1)}{1^2+25} = \frac{2}{26}$$

To find B, multiply both sides by s and take the limit as $s \to \infty$, you get the following:

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -\frac{2}{26}$$

to find C, pick a nice number, let $s=0$. then

$$0 = \frac{A}{0-1} + \frac{B(0)+C}{0^2+25} \Rightarrow \frac{C}{25} = A \Rightarrow$$

$$C = \frac{25 \cdot 2}{26}$$

then $\mathcal{L}^{-1}\left\{\frac{25}{(s-1)(s^2+25)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{2}{26}}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{2}{26}s + \frac{2 \cdot 25}{26}}{s^2+25}\right\}$

$$= \frac{2}{26}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{2}{26}\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{2 \cdot 25}{26}\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\}$$

$$= \frac{2}{26}e^{t} - \frac{2}{26} \cdot \cos 5t + \frac{2 \cdot 25}{26 \cdot 5} \cdot \sin 5t$$

(b) $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 2$

$$\mathcal{L}\{y'' + 2y' + y\} = 0 \Rightarrow \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$$

$$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = 0$$

$$\Rightarrow s^2\mathcal{L}\{y\} - s - 2 + 2s\mathcal{L}\{y\} - 2 + \mathcal{L}\{y\} = 0 \Rightarrow$$

$$\mathcal{L}\{y\}(s^2 + 2s + 1) = s + 4 \Rightarrow$$

$$\mathcal{L}\{y\} = \frac{s+4}{s^2+2s+1} = \frac{s+4}{(s+1)^2} \Rightarrow \text{Take the inverse.}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1+3}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2}\Big|_{s \to s+1}\right\}$$

$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s+3}{s^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{3}{s^2}\right\} =$

$e^{-t}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\}\right) =$

$e^{-t}(1+3t) =$

$e^{-t} + 3te^{-t}.$

(c) $y'' - 6y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1$

① Take Laplace transform of both sides of the equation.

$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t\}$, ② then use linearity

and the formulars for the transform of derivatives:

$$\boxed{\begin{aligned} \mathcal{L}\{y'\} &= s\mathcal{L}\{y\} - y(0) \\ \mathcal{L}\{y''\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) \end{aligned}}$$

$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \frac{1}{s^2} \Rightarrow$

$s^2\mathcal{L}\{y\} - \cancel{sy(0)}^{0} - \underbrace{y'(0)}_{1} - 6(s\mathcal{L}\{y\} - \cancel{y(0)}^{0}) + 9\mathcal{L}\{y\} = \frac{1}{s^2} \Rightarrow$

③ Isolate $\mathcal{L}\{y\}$:

$\mathcal{L}\{y\}(s^2 - 6s + 9) - 1 = \frac{1}{s^2} \Rightarrow$

$\mathcal{L}\{y\} = \dfrac{1 + \frac{1}{s^2}}{s^2 - 6s + 9} = \dfrac{s^2+1}{s^2(s^2-6s+9)} = \dfrac{s^2+1}{s^2(s-3)^2}$

④ finally take the inverse: $y = \mathcal{L}^{-1}\left\{\frac{s^2+1}{s^2(s-3)^2}\right\}$

using partial fractions, we have

$$\frac{s^2+1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

we can very easily find B and D using cover up method.

To find B, multiply both sides by s^2 and put $s=0$ (In your head!)

$$\frac{0^2+1}{(0-3)^2} = B \Rightarrow \boxed{B = \frac{1}{9}}$$

To find D, multiply both sides by $(s-3)^2$ and put $s=3$.

$$\frac{3^2+1}{3^2} = D \Rightarrow \boxed{D = \frac{10}{9}}$$

Note that we can't find A and C by this trick.
If you want to find A this way for example, you need to multiply both sides by s and put $s=0$
But then the term $\frac{B}{s^2}$ becomes $\frac{Bs}{s^2} = \frac{B}{s}$ which goes to ∞ as $s \to 0$.

So you should find find A, C the hard way!
you need two equations.

One easy way to get one of these equations is to multiply both sides by s and then taking the limit as $s \to \infty$. This way you don't need to plug in numbers and do a lot of algebra.

so multiply by s, let $s \to \infty$, you get

$$0 = A + C$$

we need one more equation. Let's plug in a nice number! let $s = 1$.

$$\frac{1^2+1}{1^2(1-3)^2} = \frac{A}{1} + \frac{B}{1^2} + \frac{C}{-2} + \frac{D}{4} \Rightarrow$$

$$A + \frac{1}{9} - \frac{C}{2} + \frac{1}{4} \cdot \frac{10}{9} = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow A - \frac{C}{2} = \frac{1}{2} - \frac{1}{9} - \frac{10}{(4 \times 9)} = \frac{18-4-10}{36} = \frac{4}{36} = \frac{1}{9}$$

$$\begin{cases} A + C = 0 \\ A - \frac{C}{2} = \frac{1}{9} \end{cases} \Rightarrow \begin{cases} A = -C \text{ and} \\ -C - \frac{C}{2} = -\frac{3}{2}C = \frac{1}{9} \end{cases}$$

$$\Rightarrow C = -\frac{2}{27}$$

$$\Rightarrow A = \frac{2}{27}$$

$$\text{so } y = \mathcal{L}^{-1}\left\{\frac{s^2+1}{s^2(s-3)^2}\right\} =$$

$$\mathcal{L}^{-1}\left\{\frac{2/27}{s} + \frac{1/9}{s^2} + \frac{-\frac{2}{27}}{s-3} + \frac{10/9}{(s-3)^2}\right\} =$$

$$\frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 1/9\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{10}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= \frac{2}{27} + \frac{1}{9} \cdot t - \frac{2}{27} e^{3t} + \frac{10}{9} \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \Big|_{s \to s-3} \right\}$$

<u>Note</u> $\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \Big|_{s \to s-3} \right\} = e^{3t} \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} = e^{3t} \cdot t$

$$\Rightarrow y = \frac{2}{27} + \frac{1}{9} t - \frac{2}{27} e^{3t} + \frac{10}{9} t \cdot e^{3t}$$