Math 527 - Homework 4 Solutions

Find the general solution of the differential equations using the method of judicious guessing.

1

The equation to solve is

$$y'' + 3y = x^3 - 1. \tag{1}$$

We first solve the associated homogeneous equation $y'' + 3y = 0$.

$$\lambda^2 + 3 = 0$$
$$\lambda^2 = -3$$
$$\lambda = \pm\sqrt{3}i.$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x).$$

Now we make the judicious guess

$$y_p = Ax^3 + Bx^2 + Cx + D,$$

which implies

$$y_p' = 3Ax^2 + 2Bx + C,$$
$$y_p'' = 6Ax + 2B.$$

Plugging these into (1) and rearranging gives

$$3Ax^3 + 3Bx^2 + (6A + 3C)x + (2B + 3D) = x^3 - 1.$$

From this we obtain the four equations

$$3A = 1, \qquad 3B = 0, \qquad 6A + 3C = 0, \qquad 2B + 3D = -1,$$

which imply $A = \frac{1}{3}$, $B = 0$, $C = -\frac{2}{3}$, $D = -\frac{1}{3}$. Consequently,

$$y_p = \frac{1}{3}x^3 - \frac{2}{3}x - \frac{1}{3}$$

and the general solution is

$$y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x) + \frac{1}{3}x^3 - \frac{2}{3}x - \frac{1}{3}.$$

2

The equation to solve is

$$y'' - 10y' + 25y = 30x + 3 \tag{2}$$

We first solve the associated homogeneous equation $y'' - 10y' + 25y = 0$.

$$\begin{aligned} \lambda^2 - 10\lambda + 25 &= 0 \\ (\lambda - 5)^2 &= 0 \\ \lambda &= 5 \ \text{ (repeated root).} \end{aligned}$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 e^{5x} + C_2 x e^{5x}.$$

Now we make the judicious guess

$$y_p = Ax + B,$$

which implies $y_p' = A$, $y_p'' = 0$. Plugging these into (2) and rearranging gives

$$25Ax + (-10A + 25B) = 30x + 3.$$

From this we obtain the equations

$$25A = 30, \qquad -10A + 25B = 3,$$

which imply $A = \frac{6}{5}$, $B = \frac{3}{5}$. Consequently,

$$y_p = \frac{6}{5}x + \frac{3}{5}$$

and the general solution is

$$y = C_1 e^{5x} + C_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}.$$

3

The equation to solve is

$$4y'' - 4y' - 3y = \cos(2x). \tag{3}$$

We first solve the associated homogeneous equation $4y'' - 4y' - 3y = 0$.

$$\begin{aligned} 4\lambda^2 - 4\lambda - 3 &= 0 \\ (2\lambda + 1)(2\lambda - 3) &= 0 \\ \lambda &= -\frac{1}{2}, \frac{3}{2}. \end{aligned}$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{2}x}.$$

Now we make the judicious guess

$$y_p = A\cos(2x) + B\sin(2x),$$

which implies

$$\begin{aligned} y_p' &= -2A\sin(2x) + 2B\cos(2x), \\ y_p'' &= -4A\cos(2x) - 4B\sin(2x). \end{aligned}$$

Plugging these into (3) and simplifying gives

$$(-19A - 8B)\cos(2x) + (8A - 19B)\sin(2x) = \cos(2x).$$

From this we obtain the equations

$$-19A - 8B = 1, \qquad 8A - 19B = 0,$$

which imply $A = -\frac{19}{425}$, $B = -\frac{8}{425}$. Consequently,

$$y_p = -\frac{19}{425}\cos(2x) - \frac{8}{425}\sin(2x)$$

and the general solution is

$$y = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{2}x} - \frac{19}{425}\cos(2x) - \frac{8}{425}\sin(2x).$$

4

The equation to solve is

$$y'' + 4y = 3\sin(2x). \tag{4}$$

We first solve the associated homogeneous equation $y'' + 4y = 0$.

$$\lambda^2 + 4 = 0$$

$$\lambda^2 = -4$$

$$\lambda = \pm 2i.$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 \cos(2x) + C_2 \sin(2x).$$

The obvious judicious guess, $y_p = A\cos(2x) + B\sin(2x)$, won't work because it is a solution to the associated homogeneous equation. So instead we use

$$y_p = x(A\cos(2x) + B\sin(2x)),$$

which implies

$$y_p' = A\cos(2x) + B\sin(2x) + x(-2A\sin(2x) + 2B\cos(2x)),$$

$$y_p'' = -4A\sin(2x) + 4B\cos(2x) + x(-4A\cos(2x) - 4B\sin(2x)).$$

Plugging these into (4) and simplifying gives

$$-4A\sin(2x) + 4B\cos(2x) = 3\sin(2x),$$

which implies $A = -\frac{3}{4}$, $B = 0$. Consequently,

$$y_p = -\frac{3}{4}x\cos(2x)$$

and the general solution is

$$y = C_1\cos(2x) + C_2\sin(2x) - \frac{3}{4}x\cos(2x).$$

5

The equation to solve is

$$y'' - y' + \frac{y}{4} = 3 + e^{\frac{x}{2}}.$$

Just to eliminate fractions, we'll start by multiplying both sides of the equation by 4:

$$4y'' - 4y' + y = 12 + 4e^{\frac{x}{2}}. \tag{5.1}$$

We first solve the associated homogeneous equation $4y'' - 4y' + y = 0$.

$$4\lambda^2 - 4\lambda + 1 = 0$$

$$(2\lambda - 1)^2 = 0$$

$$\lambda = \frac{1}{2} \text{ (repeated root).}$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}.$$

Since the right-hand side of (5.1) is the sum of functions from two different families, we find the particular solution in two steps. First we find a solution y_{p_1} to the equation

$$4y'' - 4y' + y = 12. \tag{5.2}$$

We make the judicious guess $y_{p_1} = A$, which implies $y'_{p_1} = y''_{p_1} = 0$. Plugging these into (5.2) gives $A = 12$, which means

$$y_{p_1} = 12.$$

Next we find a solution y_{p_2} to the equation

$$4y'' - 4y' + y = e^{\frac{x}{2}}. \tag{5.3}$$

The obvious judicious guess, $y_{p_2} = Be^{\frac{x}{2}}$, won't work because it is a solution to the associated homogeneous equation. Our second guess would be $y_{p_2} = Bxe^{\frac{x}{2}}$, but the same issue arises. So we use

$$y_{p_2} = Bx^2 e^{\frac{x}{2}},$$

which implies

$$y'_{p_2} = 2Bxe^{\frac{x}{2}} + \frac{1}{2}Bx^2 e^{\frac{x}{2}},$$

$$y''_{p_2} = 2Be^{\frac{x}{2}} + 2Bxe^{\frac{x}{2}} + \frac{1}{4}Bx^2 e^{\frac{x}{2}}.$$

Plugging these into (5.3) and simplifying gives

$$8Be^{\frac{x}{2}} = 4e^{\frac{x}{2}}$$

$$8B = 4$$

$$B = \frac{1}{2}.$$

Consequently,

$$y_{p_2} = \frac{1}{2}x^2 e^{\frac{x}{2}}$$

and the general solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x} + 12 + \frac{1}{2}x^2 e^{\frac{x}{2}}. \end{aligned}$$

6

The equation to solve is

$$y'' - 2y' + 5y = e^x \cos(2x). \tag{6}$$

We first solve the associated homogeneous equation $y'' - 2y' + 5y = 0$.

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i. \end{aligned}$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 e^x \cos(2x) + C_2 e^x \sin(2x).$$

The obvious judicious guess, $y_p = e^x(A\cos(2x) + B\sin(2x))$, won't work because it is a solution to the associated homogeneous equation. So instead we use

$$y_p = xe^x(A\cos(2x) + B\sin(2x)).$$

To simplify our calculations, let $s = A\cos(2x) + B\sin(2x)$ so that $y_p = xe^x s$. Observe that $s'' = -4s$ and we get

$$\begin{aligned} y_p' &= e^x s + xe^x s + xe^x s', \\ y_p'' &= 2e^x s + 2e^x s' - 3xe^x s + 2xe^x s'. \end{aligned}$$

Plugging these into (6) and simplifying gives

$$\begin{aligned} 2e^x s' &= e^x \cos(2x) \\ 2s' &= \cos(2x) \\ -4A\sin(2x) + 4B\cos(2x) &= \cos(2x), \end{aligned}$$

which implies $A = 0$, $B = \frac{1}{4}$. Consequently,

$$y_p = \frac{1}{4} xe^x \sin(2x)$$

and the general solution is

$$y = C_1 e^x \cos(2x) + C_2 e^x \sin(2x) + \frac{1}{4} xe^x \sin(2x).$$

7

The equation to solve is

$$y'' + 2y' + y = \sin x + 3\cos(2x). \tag{7.1}$$

We first solve the associated homogeneous equation $y'' + 2y' + y = 0$.

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda+1)^2 &= 0 \\ \lambda &= -1 \ \text{ (repeated root).} \end{aligned}$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 e^{-x} + C_2 x e^{-x}.$$

Since the two trig functions on the right-hand side of (7.1) have different arguments (one is x, the other $2x$), we find the particular solution in two steps. First we find a solution y_{p_1} to the equation

$$y'' + 2y' + y = \sin x. \tag{7.2}$$

We make the judicious guess

$$y_{p_1} = A\cos x + B\sin x,$$

which implies

$$\begin{aligned} y_{p_1}' &= -A\sin x + B\cos x, \\ y_{p_1}'' &= -A\cos x - B\sin x. \end{aligned}$$

Plugging these into (7.2) and simplifying gives

$$-2A\sin x + 2B\cos x = \sin x,$$

which implies $A = -\frac{1}{2}$, $B = 0$. Therefore,

$$y_{p_1} = -\frac{1}{2}\cos x.$$

Next we find a solution y_{p_2} to the equation

$$y'' + 2y' + y = 3\cos(2x). \tag{7.3}$$

We make the judicious guess

$$y_{p_2} = C\cos(2x) + D\sin(2x),$$

which implies

$$\begin{aligned} y_{p_2}' &= -2C\sin(2x) + 2D\cos(2x), \\ y_{p_2}'' &= -4C\cos(2x) - 4D\sin(2x). \end{aligned}$$

Plugging these into (7.3) and simplifying gives

$$(4D - 3C)\cos(2x) + (-3D - 4C)\sin(2x) = 3\cos(2x).$$

From this we obtain equations

$$4D - 3C = 3, \qquad -3D - 4C = 0,$$

which imply $C = -\frac{9}{25}$, $D = \frac{12}{25}$. Consequently,

$$y_{p_2} = -\frac{9}{25}\cos(2x) + \frac{12}{25}\sin(2x)$$

and the general solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 e^{-x} + C_2 x e^{-x} - \frac{1}{2}\cos x - \frac{9}{25}\cos(2x) + \frac{12}{25}\sin(2x). \end{aligned}$$

8

The equation to solve is

$$y'' - y = x^2 e^{3x}. \tag{8}$$

We first solve the associated homogeneous equation $y'' - y = 0$.

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1.$$

Therefore, the solution to the associated homogeneous equation is

$$y_c = C_1 e^x + C_2 e^{-x}.$$

Now we make the judicious guess

$$y_p = (Ax^2 + Bx + C)e^{3x},$$

which implies

$$y_p' = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x},$$

$$y_p'' = 2Ae^{3x} + 6(2Ax + B)e^{3x} + 9(Ax^2 + Bx + C)e^{3x}.$$

Plugging these into (8) and simplifying gives

$$8Ax^2 + (12A + 8B)x + (2A + 6B + 8C) = x^2.$$

From this we obtain the three equations

$$8A = 1, \qquad 12A + 8B, \qquad 2A + 6B + 8C = 0,$$

which imply $A = \frac{1}{8}$, $B = -\frac{3}{16}$, $C = \frac{7}{64}$. Consequently,

$$y_p = \left(\frac{1}{8}x^2 - \frac{3}{16}x + \frac{7}{64}\right) e^{3x}$$

and the general solution is

$$y = C_1 e^x + C_2 e^{-x} + \left(\frac{1}{8}x^2 - \frac{3}{16}x + \frac{7}{64}\right) e^{3x}.$$