1. $3y'' + 6y' + 2y = 0$ (10 pts)

Find the roots of the auxiliary equation $3\lambda^2 + 6\lambda + 2 = 0$

$\lambda = \frac{-6 \pm \sqrt{6^2 - 4\cdot3\cdot2}}{2\cdot3} = \frac{-3\pm\sqrt{3}}{3} = -1 \pm \frac{\sqrt{3}}{3}$

So the general solution $y = C_1 e^{(-1-\frac{\sqrt{3}}{3})x} + C_2 e^{(-1+\frac{\sqrt{3}}{3})x}$

2. $y'' + 2y' + 3y = 0$ (10 pts)

Find the roots of the auxiliary equation $\lambda^2 + 2\lambda + 3 = 0$

$\lambda = \frac{-2\pm\sqrt{2^2-4\cdot3}}{2} = -1 \pm \sqrt{2}i$

so $y = C_1 e^{(-1-\sqrt{2}i)x} + C_2 e^{(-1+\sqrt{2}i)x}$

Or it can be written as $y = e^{-x}\cdot(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x)$

3. $y'' - 6y' + 9y = 0$ (10 pts)

Find the roots of the auxiliary equation $\lambda^2 - 6y + 9 = 0$

$\lambda_{1,2} = 3$. $\therefore$ $y = C_1 e^{3x} + C_2 x e^{3x}$

4. $4y'' - 4y' + y = 0$; $y(0) = 0$, $y'(0) = 3$ (10 pts)

Find the roots of the auxiliary equation $4\lambda^2 - 4\lambda + 1 = 0$ (+3 pts)

$\lambda_{1,2} = \frac{1}{2}$, $y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$ (+2 pts)

By initial conditions, $y(0) = 0 \Rightarrow 0 = C_1$ $\therefore y = C_2 x e^{\frac{1}{2}x}$ (+2 pts)

$y' = C_2 e^{\frac{1}{2}x} + C_2 x \cdot \frac{1}{2} e^{\frac{1}{2}x}$ $y'(0) = C_2 \cdot 1 = 3$ $\therefore$ $C_2 = 3$ (+2 pts)

Then $y = 3x e^{\frac{1}{2}x}$ (+1 pts)

5. $2y'' + y' - 10y = 0$; $y(1) = 5$, $y'(1) = 2$ (10 pts)

Find the roots of the auxiliary equation $2\lambda^2 + \lambda - 10 = 0$

$\lambda = \frac{-1\pm\sqrt{1^2+4\cdot2\cdot10}}{2\cdot2} = \frac{-1\pm9}{4} = -\frac{5}{2}, 2$; or factor $(2\lambda+5)(\lambda-2) = 0$

$y = C_1 e^{-\frac{5}{2}x} + C_2 e^{2x}$ By I.C.s, $y(1) = C_1 e^{-\frac{5}{2}} + C_2 e^2 = 5$ ①

$y' = -\frac{5}{2} C_1 e^{-\frac{5}{2}x} + 2C_2 e^{2x}$, $y'(1) = -\frac{5}{2} C_1 e^{-\frac{5}{2}} + 2C_2 e^2 = 2$ ②

$2\times① - ② \Rightarrow \frac{9}{2} C_1 e^{-\frac{5}{2}} = 8 \Rightarrow C_1 = \frac{16}{9} e^{\frac{5}{2}}$

$C_2 = \frac{5 - \frac{16}{9}}{e^2} = \frac{29}{9} e^{-2}$ So $y = \frac{16}{9} e^{\frac{5}{2}(1-x)} + \frac{29}{9} e^{2(-1+x)}$

6. $y'' + 2y' + 5y = 0$; $y(0) = 0$, $y'(0) = 2$ (10 pts)

Find the roots of the auxiliary equation $\lambda^2 + 2\lambda + 5 = 0$

$(\lambda + 1)^2 + 4 = 0 \Rightarrow \lambda = -1 \pm 2i \quad \therefore\ y = e^{-x} \cdot (C_1 \cos 2x + C_2 \sin 2x)$

$y(0) = C_1 = 0$ Then $y = e^{-x} \cdot C_2 \sin 2x$, $y' = -C_2 e^{-x} \sin 2x + 2C_2 e^{-x} \cos 2x$

$y'(0) = 2C_2 = 2 \Rightarrow C_2 = 1$, $y = e^{-x} \sin 2x$

(10 pts)

7. Proof. (1) $y'' + w^2 y = 0$, we get the auxiliary Eqn. $\lambda^2 + w^2 = 0$

$\therefore\ \lambda = \pm wi$, $y = C_1 e^{iwt} + C_2 e^{-iwt}$

To show e^{iwt} and e^{-iwt} are linearly independent,

if and only if $C_1 = C_2 = 0$, we have $C_1 e^{iwt} + C_2 e^{-iwt} = 0$

Assume $C_1 \neq 0$, $\Rightarrow e^{iwt} = -\frac{C_2}{C_1} e^{-iwt}$, By Euler's formula,

$\cos wt + i \sin wt = -\frac{C_2}{C_1} [\cos wt - i \sin wt]$

which means $\frac{-C_2}{C_1} = 1$ and $\frac{C_2}{C_1} = 1$ by compare the coefficients of the real parts and the imaginary parts.

That's impossible. $\Leftrightarrow C_1 = C_2 = 0$, we have $C_1 e^{iwt} + C_2 e^{-iwt} = 0$

(2) $y = C_1 e^{iwt} + C_2 e^{-iwt}$

$= C_1 (\cos wt + i \sin wt) + C_2 (\cos wt - i \sin wt)$

$= (C_1 + C_2) \cos wt + (C_1 i - C_2 i) \sin wt$

$= \tilde{C}_1 \cos wt + \tilde{C}_2 \sin wt$, where $\tilde{C}_1 = C_1 + C_2$, $\tilde{C}_2 = C_1 i - C_2 i$

(10 pts)

8. Proof. $(\cos x + i \sin x)^n = (e^{ix})^n = e^{inx} = \cos nx + i \sin nx$

$n = 2$, $(\cos x + i \sin x)^2 = \cos^2 x - \sin^2 x + i\, 2 \sin x \cos x = \cos 2x + i \sin 2x$

so $\cos 2x = \cos^2 x - \sin^2 x$, $\sin 2x = 2 \sin x \cos 2x$

(10 pts) 9. $t\frac{d^2y}{dt^2} - (1+3t)\frac{dy}{dt} + 3y = 0$, using the ansatz $y(t) = e^{\lambda t}$

$\Rightarrow t\cdot\lambda^2 e^{\lambda t} - (1+3t)\lambda e^{\lambda t} + 3e^{\lambda t} = 0$, $[t\lambda^2 - (1+3t)\lambda + 3]e^{\lambda t} = 0$

so $t\lambda^2 - (1+3t)\lambda + 3 = 0$, $\lambda = \frac{(1+3t) \pm \sqrt{(1+3t)^2 - 4\cdot t\cdot 3}}{2\cdot t} = \frac{(1+3t)\pm(1-3t)}{2t} = 3$ or $\frac{1}{t}$

if $y = e^{3t}$, $\Rightarrow [9t - (1+3t)\cdot 3 + 3]e^{\lambda t} = 0$ ✓

when $y = e^{\frac{1}{t}\cdot t} = e^1 = e$ $\Rightarrow t\cdot 0 - (1+3t)\cdot 0 + 3e = 0$ X

so we already got one known solution $y_1 = e^{3t}$

Assume $y_2 = u(t)\cdot y_1$, transform the original Eqn into the standard form,

$\frac{d^2y}{dt^2} - (3+\frac{1}{t})y' + \frac{3}{t}y = 0$, $p(t) = -(3+\frac{1}{t})$

$u' = \frac{c_1 e^{-\int p\,dt}}{y_1^2} = \frac{c_1 e^{\int(3+\frac{1}{t})dt}}{e^{6t}} = c_1 e^{-6t}\cdot e^{3t+\ln|t|} = c_1 e^{-3t}|t|$

$u = c_1\int e^{-3t}|t|\,dt = c\int e^{-3t}t\,dt = -\frac{c}{3}\int t\,de^{-3t}$

$= -\frac{c}{3}[te^{-3t} - \int e^{-3t}dt] = -\frac{c}{3}[te^{-3t} + \frac{1}{3}e^{-3t}]$

$y_2 = u\cdot y_1 = -\frac{c}{3}[t + \frac{1}{3}]$, it satisfies the original ODE.

$\therefore y = C_1 e^{3t} + C_2(t + \frac{1}{3})$

(10 pts)
10. $t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$, using the ansatz $y(t) = t^\lambda$

$\Rightarrow t^2\cdot\lambda(\lambda-1)t^{\lambda-2} + \alpha\cdot t\cdot\lambda t^{\lambda-1} + \beta t^\lambda = 0$

$\Rightarrow [\lambda(\lambda-1) + \alpha\lambda + \beta]t^\lambda = 0$, so $\lambda^2 + (\alpha-1)\lambda + \beta = 0$ } (+4 pts)

$\lambda = \frac{1-\alpha \pm\sqrt{(\alpha-1)^2 - 4\beta}}{2}$ (assume $(\alpha-1)^2 - 4\beta > 0$)

$\lambda_1 = \frac{1-\alpha+\sqrt{(\alpha-1)^2-4\beta}}{2}$, $\lambda_2 = \frac{1-\alpha-\sqrt{(\alpha-1)^2-4\beta}}{2}$ (+4 pts)

Then $y(t) = C_1 t^{\lambda_1} + C_2 t^{\lambda_2}$ (+2 pts)