Math 527 - Homework 1 Solutions

Find the general solution of these separable ODEs. If an initial value is provided, also solve the initial value problem.

1. $\dfrac{dy}{dt} = 1 + t + y + yt$

   Solution.

$$\frac{dy}{dt} = (1+y)(1+t)$$

$$\frac{1}{1+y}\frac{dy}{dt} = 1+t \qquad \text{For } y \neq -1.$$

$$\int \frac{1}{1+y}\frac{dy}{dt}\,dt = \int (1+t)\,dt \qquad \text{Integrate both sides with respect to } t.$$

$$\int \frac{1}{1+y}\,dy = \int (1+t)\,dt$$

$$\ln(1+y) = t + \frac{t^2}{2} + C_1$$

$$1+y = e^{t+\frac{t^2}{2}+C_1} \qquad \text{Exponentiate both sides.}$$

$$1+y = e^{C_1}e^{t+\frac{t^2}{2}} \qquad C_1 \text{ is an arbitrary constant.}$$

$$1+y = C_2 e^{t+\frac{t^2}{2}} \qquad \text{Let } C_2 = e^{C_1} \text{ (so } C_2 > 0).$$

$$y = C_2 e^{t+\frac{t^2}{2}} - 1.$$

   Consider $y(t) = -1$. It is easy to verify that this is also a solution to the ODE. □

2. $\dfrac{dy}{dx} = e^{x+y+3}$

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= e^y e^{x+3} \\
e^{-y}\frac{dy}{dx} &= e^{x+3} \\
\int e^{-y}\frac{dy}{dx}\,dx &= \int e^{x+3}\,dx && \text{Integrate both sides with respect to } x. \\
\int e^{-y}\,dy &= \int e^{x+3}\,dx \\
-e^{-y} &= e^{x+3} + C_1 && C_1 \text{ is an arbitrary constant.} \\
e^{-y} &= C_2 - e^{x+3} && \text{Let } C_2 = -C_1 \text{ (so } C_2 \text{ is also arbitrary).} \\
\ln(e^{-y}) &= \ln\left(C_2 - e^{x+3}\right) && \text{Take the natural logarithm of both sides.} \\
-y &= \ln\left(C_2 - e^{x+3}\right) \\
y &= -\ln\left(C_2 - e^{x+3}\right) \\
y &= \ln\left(\frac{1}{C_2 - e^{x+3}}\right).
\end{aligned}$$

□

3. $\dfrac{dy}{dt} = \dfrac{2t}{y + yt^2}, \quad y(2) = 3$

Solution.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{2t}{y(1+t^2)} \\
y\frac{dy}{dt} &= \frac{2t}{1+t^2} \\
\int y\frac{dy}{dt}\,dt &= \int \frac{2t}{1+t^2}\,dt && \text{Integrate both sides with respect to } t. \\
\int y\,dy &= \int \frac{2t}{1+t^2}\,dt \\
\frac{y^2}{2} &= \ln(1+t^2) + C_1 && C_1 \text{ is an arbitrary constant.} \\
y^2 &= 2\ln(1+t^2) + C_2 && \text{Let } C_2 = 2C_1 \text{ (so } C_2 \text{ is also arbitrary).} \\
y &= \pm\sqrt{2\ln(1+t^2) + C_2}.
\end{aligned}$$

Plugging in the initial condition to solve for C_2, we get

$$\begin{aligned}
3 &= \pm\sqrt{2\ln 5 + C_2} \\
9 &= 2\ln 5 + C_2 \\
C_2 &= 9 - 2\ln 5.
\end{aligned}$$

Given that the y-value for our initial condition is positive, we omit $\pm$ in our answer. Therefore, our solution to the IVP is

$$\begin{aligned}
y &= \sqrt{2\ln(1+t^2) + 9 - 2\ln 5} \\
y &= \sqrt{2\ln\left(\frac{1+t^2}{5}\right) + 9}.
\end{aligned}$$

□

4. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$

Solution.

$$
\begin{aligned}
2(y-1)\frac{dy}{dx} &= 3x^2 + 4x + 2 \\
(2y-2)\frac{dy}{dx} &= 3x^2 + 4x + 2 \\
\int (2y-2)\frac{dy}{dx}\,dx &= \int (3x^2 + 4x + 2)\,dx && \text{Integrate both sides with respect to } x. \\
\int (2y-2)\,dy &= \int (3x^2 + 4x + 2)\,dx \\
y^2 - 2y &= x^3 + 2x^2 + 2x + C_1 && C_1 \text{ is an arbitrary constant.} \\
y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + C_2 && \text{Complete the square; let } C_2 = C_1 + 1. \\
(y-1)^2 &= x^3 + 2x^2 + 2x + C_2 \\
y - 1 &= \pm\sqrt{x^3 + 2x^2 + 2x + C_2} \\
y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + C_2}.
\end{aligned}
$$

Plugging in the initial condition to solve for C_2, we get

$$
\begin{aligned}
-1 &= 1 \pm \sqrt{C_2} \\
-2 &= \pm\sqrt{C_2} \\
4 &= C_2.
\end{aligned}
$$

Given that the y-value for our initial condition is negative, we replace $\pm$ with a minus sign in our answer. Therefore, our solution to the IVP is

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

□

5. $\cos y \sin t \frac{dy}{dt} = \sin y \cos t$

Solution.

$$\cot y \frac{dy}{dt} = \cot t \qquad \text{For } \sin y \neq 0,\ \sin t \neq 0.$$

$$\int \cot y \frac{dy}{dt}\, dy = \int \cot t\, dt \qquad \text{Integrate both sides with respect to } t.$$

$$\int \cot y\, dy = \int \cot t\, dt$$

$$\ln(\sin y) = \ln(\sin t) + C_1 \qquad C_1 \text{ is an arbitrary constant.}$$

$$\sin y = e^{\ln(\sin t) + C_1} \qquad \text{Exponentiate both sides.}$$

$$\sin y = e^{C_1} e^{\ln(\sin t)}$$

$$\sin y = C_2 \sin t \qquad \text{Let } C_2 = e^{C_1} \text{ (so } C_2 > 0).$$

$$y = \sin^{-1}(C_2 \sin t).$$

Consider $\sin y = 0$. Solutions to this equation are functions of the form $y(t) = k\pi$, where k is an integer. It it easy to verify that these functions also satisfy the ODE. □

Find the general solution of these 1st order linear ODEs. If an initial value is provided, also solve the initial value problem.

6. $\dfrac{dy}{dt} + y\cos t = 0$

Solution. Let $p(t) = \cos t$. Then our integrating factor is

$$\mu(t) = e^{\int \cos t\,dt} = e^{\sin t}.$$

Multiplying both sides of the ODE by $\mu(t)$, we get:

$$\begin{aligned}
\frac{dy}{dt}e^{\sin t} + ye^{\sin t}\cos t &= 0 \\
\frac{d}{dt}\left(ye^{\sin t}\right) &= 0 \\
\int \frac{d}{dt}\left(ye^{\sin t}\right)\,dt &= \int 0\,dt && \text{Integrate both sides with respect to } t. \\
ye^{\sin t} &= C && C \text{ is an arbitrary constant.} \\
y &= Ce^{-\sin t}.
\end{aligned}$$

□

7. $\dfrac{dy}{dt} - 2ty = t, \quad y(0) = 1$

Solution. Let $p(t) = -2t$. Then our integrating factor is

$$\mu(t) = e^{\int -2t\,dt} = e^{-t^2}.$$

Multiplying both sides of the ODE by $\mu(t)$, we get:

$$\begin{aligned}
\frac{dy}{dt}e^{-t^2} - 2yte^{-t^2} &= te^{-t^2} \\
\frac{d}{dt}\left(ye^{-t^2}\right) &= te^{-t^2} \\
\int \frac{d}{dt}\left(ye^{-t^2}\right)\,dt &= \int te^{-t^2}\,dt && \text{Integrate both sides with respect to } t. \\
ye^{-t^2} &= -\frac{1}{2}e^{-t^2} + C && C \text{ is an arbitrary constant.} \\
y &= Ce^{t^2} - \frac{1}{2}.
\end{aligned}$$

Plugging in the initial condition to solve for C, we get

$$\begin{aligned}
1 &= C - \frac{1}{2} \\
\frac{3}{2} &= C.
\end{aligned}$$

Therefore, our solution to the IVP is

$$y = \frac{3}{2}e^{t^2} - \frac{1}{2}.$$

□

8. $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}\, y = \dfrac{1}{1+x^2}$

Solution. Let $p(x) = \dfrac{2x}{1+x^2}$. Then our integrating factor is

$$\mu(x) = e^{\int (2x)/(1+x^2)\, dx} = e^{\ln(1+x^2)} = 1 + x^2.$$

Multiplying both sides of the ODE by $\mu(x)$, we get:

$$\begin{aligned}
\frac{dy}{dx}(1+x^2) + 2yx &= 1 \\
\frac{d}{dx}\left(y(1+x^2)\right) &= 1 \\
\int \frac{d}{dx}\left(y(1+x^2)\right)\, dx &= \int dx && \text{Integrate both sides with respect to } x. \\
y(1+x^2) &= x + C && C \text{ is an arbitrary constant.} \\
y &= \frac{x+C}{1+x^2}.
\end{aligned}$$

□

9. $\dfrac{dy}{dt} + y = te^t$

Solution. Let $p(t) = 1$. Then our integrating factor is

$$\mu(t) = e^{\int dt} = e^t.$$

Multiplying both sides of the ODE by $\mu(t)$, we get:

$$\begin{aligned}
e^t\left(\frac{dy}{dt} + y\right) &= e^t\left(te^t\right) \\
\frac{dy}{dt}e^t + ye^t &= te^{2t} \\
\frac{d}{dt}\left(ye^t\right) &= te^{2t} \\
\int \frac{d}{dt}\left(ye^t\right)\,dt &= \int te^{2t}\,dt && \text{Integrate both sides with respect to } t. \\
ye^t &= \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C && C \text{ is an arbitrary constant.} \\
y &= \frac{1}{2}te^t - \frac{1}{4}e^t + Ce^{-t}.
\end{aligned}$$

□

10. $x\dfrac{dy}{dx} - y = x^2 \sin x$

Solution. We start by dividing both sides of the equation by x to put this first order linear ODE into "standard" form:

$$\frac{dy}{dx} - \frac{1}{x}\,y = x\sin x.$$

Let $p(x) = -\frac{1}{x}$. Then our integrating factor is

$$\mu(x) = e^{\int -1/x\,dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying both sides of the "standard" form ODE by $\mu(x)$, we get:

$$\begin{aligned}
\frac{1}{x}\left(\frac{dy}{dx} - \frac{1}{x}\,y\right) &= \frac{1}{x}\,(x\sin x) \\
\frac{dy}{dx}\frac{1}{x} - \frac{1}{x^2}\,y &= \sin x \\
\frac{d}{dx}\left(y\,\frac{1}{x}\right) &= \sin x \\
\int \frac{d}{dx}\left(y\,\frac{1}{x}\right)dx &= \int \sin x\,dx && \text{Integrate both sides with respect to } x. \\
y\,\frac{1}{x} &= -\cos x + C && C \text{ is an arbitrary constant.} \\
y &= -x\cos x + Cx.
\end{aligned}$$

□