Final exam, Mar. 16, 2013 **Name:** J. Gibson
Math 527, University of New Hampshire **Section:** solutions

INSTRUCTIONS: PLEASE READ CAREFULLY

1. Write your name and section number above. Two points deducted for either if missing or illegible.
2. Always write equations.
3. Show your work and put a box or circle around your answers.
4. Final answers should be simplified as much as possible.
5. Partial credit will be given only if your work is written clearly and in equations.
6. If you have time, check your answers by differentiation and substitution.

Problem 1: (20 points) Find the general solution of the differential equation.

$$\frac{d^2y}{dt^2} + 9y = \sin(2t)$$

(2nd order const coeff nonhomog)

homog eqn $y'' + 9y = 0$. w ansatz $y = e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0$$
$$\lambda = \pm 3i$$

homog solns $y_1(t) = \cos 3t, \quad y_2(x) = \sin 3t$

ansatz for particular soln

$$y_p(t) = a\cos 2t + b\sin 2t$$
$$y_p'(t) = -2a\sin 2t + 2b\cos 2t$$
$$y_p''(t) = -4a\cos 2t - 4b\sin 2t$$

subs into nonhomog eqn

$$-4a\cos 2t - 4b\sin 2t + 9a\cos 3t + 9b\sin 2t = \sin 2t$$

coeffs of $\cos 2t \Rightarrow a = 0$

coeff of $\sin 2t \Rightarrow -4b + 9b = 1$

$$5b = 1$$
$$b = 1/5$$

particular soln

$$y_p(t) = \frac{1}{5}\sin 2t$$

genl soln

$$\boxed{y(t) = c_1\cos 3t + c_2\sin 3t + \frac{1}{5}\sin 2t}$$

Problem 2: (15 points) Find the general solution of the differential equation.

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}$$

(1st order linear)

$P(x) = -1 \Rightarrow$ integrating factor $\mu(x) = e^{\int P(x)\,dx}$

$= e^{-x}$

$$e^{-x}\frac{dy}{dx} - ye^{-x} = \frac{11}{8}e^{-4x/3}$$

$$\frac{d}{dx}\left(e^{-x}y\right) = \frac{11}{8}e^{-4x/3}$$

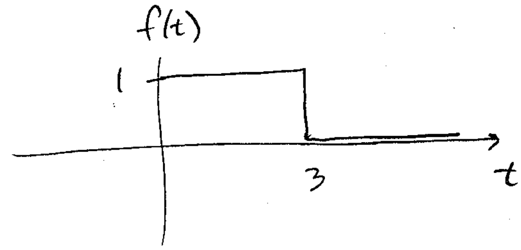
$$e^{-x}y = \frac{11}{8}\int e^{-4x/3}\,dx$$

$$= -\frac{3}{4}\cdot\frac{11}{8}e^{-4x/3} + C$$

$$\boxed{y(x) = -\frac{33}{32}e^{-x/3} + Ce^{x}}$$

Problem 3: (20 points) Find the solution of the initial value problem.

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=\underbrace{\begin{cases}1, & 0\le t<3\\ 0, & 3\le t\end{cases}}_{f(t)} \qquad y'(0)=y(0)=0.$$



$$f(t) = 1-\mathcal{U}(t-3)$$

$$y''+2y'+5y = 1-\mathcal{U}(t-3)$$

$$s^2Y(s) - s\cancel{y(0)}^{\to 0} - \cancel{y'(0)}^{\to 0} + 2sY(s) - 2\cancel{y(0)}^{\to 0} + 5Y(s) = \frac{1}{s} - e^{-3s}\frac{1}{s}$$

$$(s^2+2s+5)Y(s) = \frac{1}{s}(1-e^{-3s})$$

$$Y(s) = \frac{1}{s(s^2+2s+5)}(1-e^{-3s})$$

$$\frac{1}{s(s^2+2s+5)} = \frac{A}{s}+\frac{Bs+C}{s^2+2s+5} \qquad \text{cover-up} \Rightarrow A=\frac{1}{5}$$

$$1 = \frac{1}{5}(s^2+2s+\cancel{5}) + Bs^2+Cs$$

$$0 = \frac{1}{5}s^2+\frac{2}{5}s + Bs^2+Cs \quad \Rightarrow \quad B=-\frac{1}{5}, \; C=-\frac{2}{5}$$

$$Y(s) = \frac{1}{5}(1-e^{-3s})\left[\frac{1}{s}+\frac{-s-2}{s^2+2s+5}\right]$$

$$Y(s) = \frac{1}{5}\left[\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right] - \frac{1}{5}e^{-3s}\left[\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right]$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{e^{-3s}\left[\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right]\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right\} - \frac{1}{5}\mathcal{U}(t-3)\left[\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right\}\right]_{t\to t-3}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+4}\right\}$$

$$= 1 - \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\Big|_{s\to s+1}\right\}$$

$$= 1 - e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} - \frac{1}{2}\,\frac{2}{s^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s-2}{s^2+2s+5}\right\} = 1 - e^{-t}\left[\cos 2t - \frac{1}{2}\sin 2t\right]$$

so

$$y(t) = \frac{1}{5} - \frac{1}{5}e^{-t}\left(\cos 2t - \frac{1}{2}\sin 2t\right) - \frac{1}{5}\mathcal{U}(t-3)\left[1 - e^{-t}\left(\cos 2t - \frac{1}{2}\sin 2t\right)\right]_{t \to t-3}$$

$$\boxed{y(t) = \frac{1}{5}\left[1 - e^{-t}\left(\cos 2t - \frac{1}{2}\sin 2t\right)\right] - \frac{1}{5}\mathcal{U}(t-3)\left[1 - e^{-(t-3)}\left(\cos 2(t-3) - \frac{1}{2}\sin 2(t-3)\right)\right]}$$

Problem 4: (15 points) Find the general solution of the differential equation as a power series centered about $x = 0$. Simplify your answer as much as you can.

$$(x-3)\frac{dy}{dx} + 2y = 0$$

ansatz $\quad y(x) = \sum_{n=0}^{\infty} c_n x^n$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$(x-3)\sum_{n=0}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} n c_n x^n - \sum_{\cancel{n=0}\atop n=1}^{\infty} 3 n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2) c_n x^n - \sum_{m=0}^{\infty} 3(m+1) c_{m+1} x^m = 0$$

$$\left(\text{letting } m = n-1 \atop m+1 = n\right)$$

$$\sum_{n=0}^{\infty} (n+2) c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)c_n - 3(n+1)c_{n+1}\right] x^n = 0$$

$$\Rightarrow (n+2)c_n - 3(n+1)c_{n+1} = 0 \qquad \text{for } n = 0, 1, 2, \ldots$$

$$c_{n+1} = \frac{n+2}{3(n+1)} c_n$$

let $c_0 = 1$

$$c_1 = \frac{2}{3} c_0 = \frac{2}{3}$$

$$c_2 = \frac{3}{3\cdot 2} c_1 = \frac{3}{3\cdot\cancel{2}}\,\frac{\cancel{2}}{3} = \frac{3}{3^2}$$

$$c_3 = \frac{4}{3\cdot 3} c_2 = \frac{4}{3\cdot\cancel{3}}\,\frac{\cancel{3}}{3^2} = \frac{4}{3^3}$$

$$c_4 = \frac{5}{3\cdot 4} c_3 = \frac{5}{3\cdot\cancel{4}}\,\frac{\cancel{4}}{3^3} = \frac{5}{3^4} \qquad \text{etc.}$$

$$y(x) = c\left(1 + \frac{2}{3}x + \frac{3}{3^2}x^2 + \frac{4}{3^3}x^3 + \frac{5}{3^4}x^4 + \ldots\right)$$

$$\boxed{y(x) = c\sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n}$$

Problem 5: (15 points) Find the general solution of the differential equation. Express your answer in terms of real-valued functions.

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}$$

$\underline{x}' = A\underline{x} \qquad A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (-3-\lambda)(-1-\lambda) - (-1)\cdot 2 = 0$$

$$(\lambda+3)(\lambda+1) + 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= -2 \pm \tfrac{1}{2}\sqrt{-4}$$

$$\lambda = -2 \pm i$$

$(A - \lambda I)\underline{v} = \underline{0} \quad$ for $\lambda = -2+i$

$$\begin{pmatrix} -3-(-2+i) & 2 \\ -1 & -1-(-2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

top eqn $\quad (-1-i)v_1 + 2v_2 = 0,$

$2v_2 = (1+i)v_1 \quad$ let $v_1 = 2$, then $v_2 = 1+i$

So for $\lambda = -2 \pm i$, $\quad \underline{v} = \begin{pmatrix} 2 \\ 1 \pm i \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{real part}} \pm \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{imag part}} i$

gen'l soln

$$\underline{x}(t) = c_1 e^{(-2+i)t}\begin{pmatrix} 2 \\ 1+i \end{pmatrix} + c_2 e^{(-2-i)t}\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

complex form

$$\boxed{\underline{x}(t) = c_1\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right]e^{-2t} + c_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t\right]e^{-2t}}$$

real-valued form

note: your solution might look different but still be correct, depending on how you chose to solve for the components $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of the eigenvector $\underline{v}$

Problem 6: (10 points) Read the questions carefully and answer them. Please do not ask for any kind of help on these questions during the exam.

(a) For what region of x will the series solution of Problem 4 converge?

at least $-3 < x < 3$

(b) True or false: The general solution of a second-order nonlinear differential equation is an arbitrary linear combination of its two linearly independent solutions.

false - that's true of linear equations, not nonlinear equations

(c) Write down the general solutions of the following differential equations. You do not need to show your work.

$\frac{dy}{dx} + ay = 0$ $y(x) = c e^{-ax}$

$\frac{d^2y}{dx^2} - k^2 y = 0$ $y(x) = c_1 e^{kx} + c_2 e^{-kx}$

$\frac{d^2y}{dx^2} + k^2 y = 0$ $y(x) = c_1 \cos kx + c_2 \sin kx$