Sample exam #3, Laplace transforms **Name:** J. Gibson
Math 527, University of New Hampshire **Section:** —

INSTRUCTIONS: PLEASE READ CAREFULLY

1. Write your name and section number above. Two points deducted if either is missing or illegible.
2. Show your work and put a box or circle around your answers.
3. Always write equations.
4. Final answers should be simplified as much as possible.
5. Partial credit will be given only if your work is written clearly and in equations.
6. If you have time, check your answers by differentiation and substitution.

Problem 1. (30 pts) Compute the Laplace transform or inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{e^{-as}\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{U}(t-a)\left[\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}\right]_{t\to t-a}$

$$\boxed{\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s^4}\right\} = \frac{1}{6}\mathcal{U}(t-a)\,(t-a)^3}$$

method of coverup

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \Rightarrow A = -1/2,\ B = 1/2$$

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^2-2s}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$$= -\frac{1}{2}\cdot 1 + \frac{1}{2}e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s}\right\} = \frac{1}{2}\left[e^{2t}-1\right]}$$

(c) $\mathscr{L}\{te^{-3t}\sin 2t\} = \left[\mathcal{L}\{t\sin 2t\}\right]_{s\to s+3}$

$$= \left[-\frac{d}{ds}\mathcal{L}\{\sin 2t\}\right]_{s\to s+3}$$

$$= \left[-\frac{d}{ds}\frac{2}{s^2+4}\right]_{s\to s+3}$$

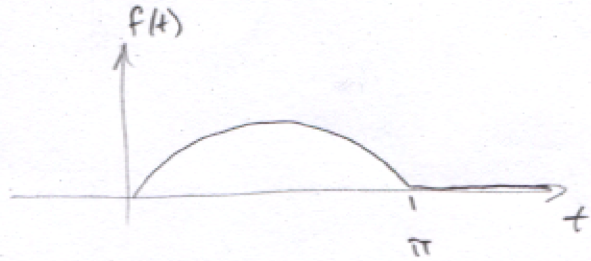
$$= \left[-\frac{0-4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \frac{4(s+3)}{((s+3)^2+4)^2}$$

$$\boxed{\mathcal{L}\{te^{-3t}\sin 2t\} = \frac{4s+12}{(s^2+6s+13)^2}}$$

Problem 2. (30 pts) Express $f(t)$ in terms of Heaviside functions and then compute $\mathscr{L}\{f(t)\}$.

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t \end{cases}$$



$$= \sin t \begin{cases} 1 & 0 \le t < \pi \\ 0 & \pi \le t \end{cases}$$

$$= (1 - \mathcal{U}(t-\pi)) \sin t$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{\mathcal{U}(t-\pi)\sin t\}$$

$$= \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$$

$\sin(t+\pi) = -\sin t$

$$= \frac{1}{s^2+1} + e^{-\pi s}\mathcal{L}\{\sin t\}$$

$$\boxed{\mathcal{L}\{f(t)\} = \left(1 + e^{-\pi s}\right)\frac{1}{s^2+1}}$$

Problem 3. (40 pts) Find the solution of the initial value problem using Laplace transforms. Derivatives y', y'' are with respect to t.

$$y'' + 4y' + 8y = e^{-t}, \quad y(0) = 0, \; y'(0) = 1$$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - y'(0)^{1} + 4\left(sY(s) - \cancel{y(0)}^{0}\right) + 8Y(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 8)Y(s) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{1}{(s+1)(s^2+4s+8)} + \frac{1}{s^2+4s+8}$$

$$\frac{1}{(s+1)(s^2+4s+8)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8} \qquad \text{cover up} \Rightarrow A = 1/5$$

$$1 = \frac{1}{5}(s^2+4s+8) + (Bs+C)(s+1)$$

$$1 = \frac{1}{5}s^2 + \frac{4}{5}s + \frac{8}{5} + Bs^2 + (B+C)s + C \quad \Rightarrow B = -1/5$$

$$C = -3/5$$

$$Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s+3}{s^2+4s+8} + \frac{1}{s^2+4s+8}$$

$$= \frac{1}{5}\left[\frac{1}{s+1} - \frac{s-2}{s^2+4s+8}\right]$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{(s+2)-4}{(s+2)^2+4}\right\}$$

$$= \frac{1}{5}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{5}e^{-2t}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\left(\cos 2t - 2\sin 2t\right)$$

$$\boxed{y(t) = \frac{1}{5}e^{-t} + \frac{1}{5}e^{-2t}\left(2\sin 2t - \cos 2t\right)}$$

Extra credit. (10 pts) Solve the integral equation.

$$y(t) = t - e^t \int_0^t y(\tau)\, e^{-\tau} d\tau$$

$$= t - \int_0^t y(\tau)\, e^{-(t-\tau)} d\tau$$

$$= t - y(t) * e^{-t}$$

$$Y(s) = \mathcal{L}\{t\} - \mathcal{L}\{y(t) * e^{-t}\}$$

$$= \frac{1}{s^2} - \mathcal{L}\{y(t)\} \cdot \mathcal{L}\{e^{-t}\}$$

$$Y(s) = \frac{1}{s^2} - Y(s)\frac{1}{s+1}$$

$$\left(1 + \frac{1}{s+1}\right) Y(s) = \frac{1}{s^2}$$

$$\frac{s+2}{s+1} Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{s+1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$ coverup $\Rightarrow B = 1/2 \quad C = -1/4$

$$s+1 = As(s+2) + \tfrac{1}{2}(s+2) - \tfrac{1}{4}s^2 \Rightarrow A = 1/4$$

$$s+\cancel{1} = \tfrac{1}{4}s^2 + \tfrac{1}{2}s + \tfrac{1}{2}s + \cancel{1} - \tfrac{1}{4}s^2 \quad \checkmark$$

$$y(t) = \tfrac{1}{4}\mathcal{L}^{-1}\left\{\tfrac{1}{s}\right\} + \tfrac{1}{2}\mathcal{L}^{-1}\left\{\tfrac{1}{s^2}\right\} - \tfrac{1}{4}\mathcal{L}^{-1}\left\{\tfrac{1}{s+2}\right\}$$

$$\boxed{y(t) = \frac{1}{4} + \frac{1}{2}t - \frac{1}{4}e^{-2t}}$$