Exam #2, Oct 7, 2011
Math 527, University of New Hampshire

Name: Solutions
Section:

INSTRUCTIONS: PLEASE READ CAREFULLY
Write your name and section number above. 5 pts will deducted if either is missing or illegible.
Write your final answers in the space provided. Show your work on attached sheets. Staple together in the upper-left corner.

Problem 1 (20 pts): **DO NOT SOLVE THE DIFFERENTIAL EQUATION.**
Just give an appropriate guess for the particular solution of the nonhomogeneous equation.

(a) $y'' - 4y' + 4y = \cos 2x$ $\quad$ $y_p = A\sin(2x) + B\cos(2x)$

(b) $y'' - 4y' + 4y = e^{2x}$ $\quad$ $y_p = Ax^2e^{2x}$

(c) $y'' + 4y = \cos 2x$ $\quad$ $y_p = x(A\sin(2x) + B\cos(2x))$

(d) $y'' + 4y = x^2 + e^x \cos 2x$ $\quad$ $y_p = (Ax^2 + Bx + C) + e^x(D\sin(2x) + E\cos(2x))$

Problem 2 (30 pts): Find the general solution of the ODE

$$y'' + 2y' + 4y = 3\cos x$$

$y(x) = e^{-x}[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)] + \frac{6}{13}\sin(x) + \frac{9}{13}\cos(x)$

Problem 3 (30 pts): Find the general solution of the ODE

$$y'' + 4y' + 4y = x^{-2}e^{-2x}$$

$y(x) = c_1e^{-2x} + c_2xe^{-2x} - \ln|x|e^{-2x} - e^{-2x}$

Problem 4 (20 pts): Consider the forced mass-spring-dashpot ODE with $m > 0$, $k > 0$, and $\beta \geq 0$:

$$my'' + \beta y' + ky = f(t)$$

(a) If $\beta = 0$ and $f(t) = 0$, what is the frequency of oscillation ω? $\quad$ $\omega = \sqrt{k/m}$

(b) If $\beta = 0$, give a simple bounded function $f(t)$ that will cause unbounded growth in $y(t)$ as $t \to \infty$.

$f(t) = \alpha\sin(\sqrt{k/m}\,t) + \beta\cos(\sqrt{k/m}\,t)$
for $\alpha, \beta \in \mathbb{R}$

(c) Will the same $f(t)$ cause unbounded growth if β is increased slightly from zero? Why or why not?

No; if $\beta \neq 0$, then $f(t)$ will no longer be part of the homogeneous solution, so the guess will no longer need to be multiplied by t, and hence no longer exhibit unbounded growth.

1.) (a.) $y'' - 4y' + 4y = \cos(2x)$

Homogeneous: $y'' - 4y' + 4y = 0$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2 \Rightarrow y_{gen} = c_1 e^{2x} + c_2 x e^{2x}$$

$$y_p = A\sin(2x) + B\cos(2x)$$

(b.) $y'' - 4y' + 4y = e^{2x}$

From above, $y_{gen} = c_1 e^{2x} + c_2 x e^{2x}$

So $y_p = A x^2 e^{2x}$

(c.) $y'' + 4y = \cos(2x)$

Homogeneous: $y'' + 4y = 0$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i \Rightarrow y_{gen} = c_1 \cos(2x) + c_2 \sin(2x)$$

So $y_p = x(A\sin(2x) + B\cos(2x))$

(d.) $y'' + 4y = x^2 + e^x \cos(2x)$

From above, $y_{gen} = c_1 \cos(2x) + c_2 \sin(2x)$

So $y_p = (Ax^2 + Bx + C) + e^x[D\sin(2x) + E\cos(2x)]$

2.) $y'' + 2y' + 4y = 3\cos(x)$

Homogeneous: $y'' + 2y' + 4y = 0$

$\lambda^2 + 2\lambda + 4 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm i\sqrt{3} \Rightarrow y_{gen} = e^{-x}\left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]$

So $y_p = A\sin(x) + B\cos(x)$

$y_p' = A\cos(x) - B\sin(x)$

$y_p'' = -A\sin(x) - B\cos(x)$

Plugging into our original. non-homogeneous ODE yields:

$(-A\sin(x) - B\cos(x)) + 2(A\cos(x) - B\sin(x)) + 4(A\sin(x) + B\cos(x)) = 3\cos(x)$

$3A\sin(x) + 2A\cos(x) + 3B\cos(x) - 2B\sin(x) = 3\cos(x)$

(cos(x) terms): $2A + 3B = 3 \Rightarrow 6A + 9B = 9$

(sin(x) terms): $3A - 2B = 0 \quad -6A + 4B = 0$

$13B = 9 \quad B = 9/13 \quad A = 6/13$

So our most general solution is:

$$y(x) = e^{-x}\left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right] + 6/13\sin(x) + 9/13\cos(x).$$

3.) $y'' + 4y' + 4y = x^{-2} e^{-2x}$

Because x^{-2} is not an exponential, polynomial, or sine or cosine, we can't use judicious guessing (method of undetermined coefficients).

Variation of parameters: $y_p = u_1 y_1 + u_2 y_2$; $u_1 = \int \frac{-y_2 f(x)}{W} dx$; $u_2 = \int \frac{y_1 f(x)}{W} dx$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

First, solve the homogeneous problem:

$$y'' + 4y' + 4y = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = 0$$
$$(\lambda + 2)^2 = 0$$
$$\lambda = -2, -2 \Rightarrow y_{gen} = c_1 e^{-2x} + c_2 x e^{-2x}$$

So $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$

and $$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & -2xe^{-2x} + e^{-2x} \end{vmatrix} = -2xe^{-4x} + e^{-4x} + (2xe^{-4x}) = e^{-4x}$$

So $$u_1 = \int \frac{-xe^{-2x}(x^{-2}e^{-2x})}{e^{-4x}} dx = -\int \frac{1}{x} dx = -\ln|x|$$

and $$u_2 = \int \frac{e^{-2x}(x^{-2}e^{-2x})}{e^{-4x}} dx = \int x^{-2} dx = -\frac{1}{x}$$

So $$y_p = -\ln|x| e^{-2x} - \frac{1}{x}(xe^{-2x}) = -\ln|x| e^{-2x} - e^{-2x}$$

And our most general solution is:

$$y(x) = -\ln|x| e^{-2x} - e^{-2x} + c_1 e^{-2x} + c_2 x e^{-2x}$$

4.) $my'' + \beta y' + ky = f(t)$

(a.) Suppose $\beta = 0$, $f(t) = 0$,

$$y'' + k/m\, y = 0 \Rightarrow \lambda^2 + k/m = 0$$

$$\lambda^2 = -k/m$$

$$\lambda = \sqrt{k/m}\, i \Rightarrow y_{gen} = c_1 \cos(\sqrt{k/m}\, t) + c_2 \sin(\sqrt{k/m}\, t)$$

So $\omega = \sqrt{k/m}$.

(b.) if $f(t)$ is part of the homogeneous solution (i.e. $f(t) = \sin(\sqrt{k/m}\, t)$), then our particular solution would include $t \cdot (A \sin(\sqrt{k/m}\, t) + B \cos(\sqrt{k/m}\, t))$, which would cause unbounded growth.

(c.) No, because $f(t)$ will no longer be part of the homogeneous solution, so the guess y_p will no longer include the extra instance of t, hence no longer exhibit unbounded growth.