1.) $\frac{dy}{dx} = \frac{-y + \sin(x)}{x}$, $y(\pi) = 1$

As an EXACT equation:

$$x \frac{dy}{dx} = -y + \sin(x)$$

$$y - \sin(x) + x \frac{dy}{dx} = 0$$

$M(x,y) = y - \sin(x)$ $\qquad \frac{\partial M}{\partial y} = 1$ $\qquad$ Thus this ODE

$N(x,y) = x$ $\qquad \frac{\partial N}{\partial x} = 1$ $\qquad$ is exact.

So there exists an $f(x,y)$ such that $\partial f/\partial x = M(x,y)$, $\partial f/\partial y = N(x,y)$.

$$f(x,y) = \int M(x,y)\,dx + g(y)$$
$$= \int (y - \sin(x))\,dx + g(y)$$
$$f(x,y) = xy + \cos(x) + g(y)$$

and

$$\partial f/\partial y = x + g'(y) \overset{\text{set}}{=} x$$

$g'(y) = 0 \Rightarrow g(y) = 0$. So $f(x,y) = xy + \cos(x)$.

So $(y - \sin(x)) + x\, dy/dx = 0$

$$\partial f/\partial x + \partial f/\partial y \; dy/dx = 0$$

$\frac{d}{dx}[f(x,y)] = 0$ $\qquad$ Integrating both sides w.r.t. x yields:

$$xy + \cos(x) = c$$
$$xy = c - \cos(x)$$

$y = \frac{c - \cos(x)}{x}$ $\qquad$ plugging in $y(\pi) = 1$

$$1 = \frac{c - \cos(\pi)}{\pi}$$
$$\pi = c - (-1)$$
$$c = \pi - 1$$

So our solution is $y = \frac{(\pi - 1) - \cos(x)}{x}$.

1.) $\frac{dy}{dx} = \frac{-y + \sin(x)}{x}$, $y(\pi) = 1$

As a 1st order, linear ODE:

$$dy/dx = -y/x + \sin(x)/x$$

$$dy/dx + y/x = \sin(x)/x$$ Using integrating factors w/ $p(x) = \frac{1}{x}$

$$\mu(x) = e^{\int 1/x \, dx} = e^{\ln|x|} = x$$

$$x\left(dy/dx + y/x\right) = \left(\sin(x)/x\right) x$$

$$x \, dy/dx + y = \sin(x)$$

$$\frac{d}{dx}(xy) = \sin(x)$$ Now integrate both sides w.r.t. x

$$xy = \int \sin(x) \, dx$$

$$xy = -\cos(x) + c$$

$$y = \frac{c - \cos(x)}{x}$$ plugging in $y(\pi) = 1$

$$1 = \frac{c - (-1)}{\pi}$$

$$\pi = c + 1$$

$$c = \pi - 1$$

$$y = \frac{(\pi - 1) - \cos(x)}{x}$$

2.) $\frac{dy}{dx} = -\frac{3y^2+2y}{6xy+2x+6}$

This is an EXACT equation:

$(3y^2+2y) + (6xy+2x+6)\frac{dy}{dx} = 0$

$M(x,y) = 3y^2+2y$ $\quad \partial M/\partial y = 6y+2$ $\quad$ This ODE is

$N(x,y) = 6xy+2x+6$ $\quad \partial N/\partial x = 6y+2$ $\quad$ exact!

So there exists an $f(x,y)$ such that $\partial f/\partial x = M(x,y)$ and $\partial f/\partial y = N(x,y)$.

$$f(x,y) = \int M(x,y)\,dx + g(y)$$
$$= \int (3y^2+2y)\,dx + g(y)$$
$$f(x,y) = 3xy^2 + 2xy + g(y)$$

and

$$\partial f/\partial y = 6xy + 2x + g'(y) = 6xy + 2x + 6$$
$$g'(y) = 6 \Rightarrow g(y) = 6y$$

So $f(x,y) = 3xy^2 + 2xy + 6y$

Thus $(3y^2+2y) + (6xy+2x+6)\frac{dy}{dx} = 0$

$$\frac{d}{dx}\left[3xy^2 + 2xy + 6y\right] = 0$$ Integrating both sides w.r.t x yields:

$$3xy^2 + 2xy + 6y = c$$ Implicit sol'n

$$y^2 + 2\left(\frac{x+3}{3x}\right)y = c/3x$$

$$y^2 + 2\left(\frac{x+3}{3x}\right)y + \left(\frac{x+3}{3x}\right)^2 = \frac{c}{3x} + \left(\frac{x+3}{3x}\right)^2$$

$$\left(y + \left(\frac{x+3}{3x}\right)\right)^2 = \frac{c}{3x} + \left(\frac{x+3}{3x}\right)^2$$

$$y = \left(\pm\sqrt{\frac{c}{3x} + \left(\frac{x+3}{3x}\right)^2}\right) - \left(\frac{x+3}{3x}\right)$$ explicit sol'n

3.) $\frac{dy}{dt} - t(y^2+1) = 0, \quad y(0) = 1$

This ODE is SEPARABLE:

$$\frac{dy}{dt} = t(y^2+1)$$

$$\left(\frac{1}{y^2+1}\right)\frac{dy}{dt} = t$$

$$\frac{d}{dt}\left[\tan^{-1}(y)\right] = t \qquad \text{Integrating w.r.t. } t:$$

$$\tan^{-1}(y) = \tfrac{1}{2}t^2 + c \qquad \text{Using } y(0) = 1$$

$$\tan^{-1}(1) = \tfrac{1}{2}(0) + c$$

$$\pi/4 = c$$

Our gen'l sol'n is $y = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4}\right)$

Bonus: (a.) what is the gen'l form of a 1st order Bernoulli ODE?

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y^n$$

$$\frac{1}{1-n} - \frac{1-n}{1-n}$$

(b.) What is the appropriate substitution?

$$u = y^{1-n} \Rightarrow y = u^{\left(\frac{1}{1-n}\right)} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{1-n}\right)u^{\left(\frac{n}{1-n}\right)}\frac{du}{dx}$$

(c.) Plug the substitution into the ODE and show it results in a first order ODE:

$$\left(\frac{1}{1-n}\right)u^{\left(\frac{n}{1-n}\right)}\frac{du}{dx} + P(x)\,u^{\left(\frac{1}{1-n}\right)} = f(x)\left(u^{\left(\frac{1}{1-n}\right)}\right)^n$$

$$= f(x)\left(u^{\left(\frac{n}{1-n}\right)}\right)$$

Mult. both sides by $(1-n)\,u^{\left(\frac{n-1}{n}\right)}$

$$du/dx + P(x)(1-n)\,u = f(x)(1-n)$$