[Name]
Section [#]
Homework #2
Math 527

1. Rewrite in differential form:

$$\begin{aligned} 2x + (2y-2)\frac{dy}{dx} &= -3 \\ 2x + 3 + (2y-2)\frac{dy}{dx} &= 0 \\ (2x+3)\,dx + (2y-2)\,dy &= 0 \end{aligned}$$

This differential **is exact** by the exactness criterion:

$$\begin{aligned} \frac{\partial}{\partial y}(2x+3) &= 0; \\ \frac{\partial}{\partial x}(2y-2) &= 0. \end{aligned}$$

We want f such that $df(x,y) = (2x+3)\,dx + (2y-2)\,dy$, that is

$$\begin{array}{ccc} f_x(x,y) = 2x+3 & \text{and} & f_y(x,y) = 2y-2 \\ f(x,y) = x^2 + 3x + g(y) & & \vdots \\ f_y(x,y) = g'(y) & \text{and} & f_y(x,y) = 2y-2 \end{array} \tag{1}$$

Combine the last pair of equations:

$$\begin{aligned} g'(y) &= 2y-2 \\ g(y) &= y^2 - 2y \end{aligned} \tag{2}$$

Substitute (2) into equation (1):

$$f(x,y) = x^2 + 3x + y^2 - 2y$$

The general solution is

$$x^2 + 3x + y^2 - 2y = C$$

...which can be made explicit:

$$\begin{aligned} y^2 - 2y &= -x^2 - 3x + C \\ y^2 - 2y + 1 &= -x^2 - 3x + C \quad (C := C+1) \\ (y-1)^2 &= C - x^2 - 3x + C \\ y = 1 &= \pm\sqrt{-x^2 - 3x + C} \\ y &= 1 \pm \sqrt{-x^2 - 3x + C} \end{aligned}$$

2. Rewrite in differential form:

$$\begin{aligned} \frac{dy}{dx} &= \frac{5x+4y}{8y^3-4x} \\ \left(8y^3-4x\right)\frac{dy}{dx} &= 5x+4y \\ \left(-5x-4y\right)+\left(8y^3-4x\right)\frac{dy}{dx} &= 0 \\ \left(-5x-4y\right)dx+\left(8y^3-4x\right)dy &= 0 \end{aligned}$$

This differential **is exact:**

$$\begin{aligned} \frac{\partial}{\partial y}\left(-5x-4y\right) &= -4; \\ \frac{\partial}{\partial x}\left(8y^3-4x\right) &= -4. \end{aligned}$$

We want f such that $df\left(x,y\right)=(2x+3)\,dx+\left(2y-2\right)dy$, that is

$$\begin{array}{rclcrcl} f_x\left(x,y\right) &=& -5x-4y & \text{and} & f_y\left(x,y\right) &=& 8y^3-4x \\ \vdots & & & & f\left(x,y\right) &=& 2y^4-4xy+g\left(x\right) \\ f_x\left(x,y\right) &=& -5x-4y & \text{and} & f_x\left(x,y\right) &=& -4y+g'\left(x\right) \end{array} \qquad (3)$$

Combine the last pair of equations:

$$\begin{aligned} -4y+g'\left(x\right) &= -5x-4y \\ g'\left(x\right) &= -5x \\ g\left(y\right) &= -\frac{5}{2}x^2 \end{aligned} \qquad (4)$$

Substitute (4) into (3):

$$f\left(x,y\right)=2y^4-4xy-\frac{5}{2}x^2$$

The general solution is

$$2y^4-4xy-\frac{5}{2}x^2=C$$

...which will be left implicit.

3. Rewrite in differential form:

$$\begin{aligned} e^x\sin y-2y\sin x+\left(e^x\cos y+2\cos x\right)\frac{dy}{dx} &= 0 \\ \left(e^x\sin y-2y\sin x\right)dx+\left(e^x\cos y+2\cos x\right)dy &= 0 \end{aligned}$$

This differential **is exact:**

$$\begin{aligned} \frac{\partial}{\partial y}\left(e^x\sin y+2y\sin x\right) &= e^x\cos y+2\sin x; \\ \frac{\partial}{\partial x}\left(e^x\cos y-2\cos x\right) &= e^x\cos+2\sin x. \end{aligned}$$

We want f such that $df(x,y) = \left(e^x \sin y - 2y\sin x\right)dx + \left(e^x \cos y + 2\cos x\right)dy$, that is

$$\begin{aligned} f_x(x,y) &= e^x \sin y + 2y \sin x & \text{and} \quad f_y(x,y) &= e^x \cos y + 2\cos x \\ f(x,y) &= e^x \sin y - 2y\cos x + g(y) & &\vdots \\ f_y(x,y) &= e^x \cos y - 2\cos x + g'(y) & \text{and} \quad f_x(x,y) &= e^x \cos y + 2\cos x \end{aligned} \tag{5}$$

Combine the last pair of equations:

$$\begin{aligned} e^x \cos y - 2\cos x + g'(y) &= e^x \cos y + 2\cos x \\ g'(y) &= 0 \\ g(y) &= 0 \end{aligned} \tag{6}$$

Substitute (6) into (5):

$$f(x,y) = e^x \sin y - 2y\cos x$$

The general solution is

$$e^x \sin y - 2y\cos x = C$$

...which will be left implicit.

4. Rewrite in differential form:

$$\begin{aligned} x\ln y + xy + \left(y\ln x + xy\right)\frac{dy}{dx} &= 0 \\ \left(x\ln y + xy\right)dx + \left(y\ln x + xy\right)dy &= 0 \end{aligned}$$

This differential **is not exact:**

$$\begin{aligned} \frac{\partial}{\partial y}\left(x\ln y + xy\right)dx &= \frac{x}{y} + x; \\ \frac{\partial}{\partial x}\left(y\ln x + xy\right) &= \frac{y}{x} + y. \end{aligned}$$

(These two are not equal, for example, when $x = 1$ and $y = 2$.)

5. Rewrite in differential form:

$$\begin{aligned} x - y^3 + y^2\sin x + \left(-3xy^2 - 2y\cos x\right)\frac{dy}{dx} &= 0 \\ \left(x - y^3 + y^2\sin x\right)dx + \left(-3xy^2 - 2y\cos x\right)dy &= 0 \end{aligned}$$

This differential **is exact:**

$$\begin{aligned} \frac{\partial}{\partial y}\left(x - y^3 + y^2\sin x\right) &= -3y^2 + 2y\sin x; \\ \frac{\partial}{\partial x}\left(-3xy^2 - 2y\cos x\right) &= -3y^2 + 2y\sin x. \end{aligned}$$

We want f such that $df(x,y) = \left(x - y^3 + y^2\sin x\right)dx + \left(-3xy^2 - 2y\cos x\right)dy$, that is

$$\begin{array}{rclcrcl} f_x(x,y) &=& x - y^3 + y^2\sin x & \text{and} & f_y(x,y) &=& -3xy^2 - 2y\cos x \\ f(x,y) &=& \frac{1}{2}x^2 - xy^3 - y^2\cos x + g(y) & & & \vdots & \\ f_y(x,y) &=& -3xy^2 - 2y\cos x + g'(y) & \text{and} & f_y(x,y) &=& -3xy^2 - 2y\cos x \end{array} \tag{7}$$

Combine the last two equations:

$$\begin{aligned} -3xy^2 - 2y\cos x + g'(y) &= -3xy^2 - 2y\cos x \\ g'(y) &= 0 \\ g(y) &= 0 \end{aligned} \tag{8}$$

Substitute (8) into (7):

$$f(x,y) = \frac{1}{2}x^2 - xy^3 - y^2\cos x$$

The general solution is

$$\frac{1}{2}x^2 - xy^3 - y^2\cos x = C$$

...which will be left implicit.

6. Set $u = \frac{y}{x}$. Then

$$\begin{aligned} y &= xu; \\ \frac{dy}{dx} &= \frac{d}{dx}(xu) \\ &= u + x\frac{du}{dx}. \end{aligned}$$

Make these substitutions in the given equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2 + 2y}{y^2} \\ u + x\frac{du}{dx} &= \frac{(xu)^2 + 2x(xu)}{(xu)^2} \end{aligned}$$

Now simplify and separate:

$$\begin{aligned}
u + x\frac{du}{dx} &= \frac{x^2u^2 + 2x^2u}{x^2u^2} \\
&= \frac{2+u}{u} \\
x\frac{du}{dx} &= \frac{2+u-u^2}{u} \\
\frac{du}{dx} &= \frac{2+u-u^2}{xu} \\
\frac{u}{2+u-u^2}\frac{du}{dx} &= \frac{1}{x} \\
\frac{u}{2+u-u^2}du &= \frac{1}{x}dx
\end{aligned}$$

(Either of the last two equations is an acceptable final answer.)

7. Rewrite in standard Bernoulli form:

$$\frac{dy}{dx} + y = xy^4 \tag{9}$$

Set $u = y^{-3}$. Then

$$\begin{aligned}
y &= u^{-1/3}; \\
\frac{dy}{dx} &= -\frac{1}{3}u^{-4/3}\frac{du}{dx}.
\end{aligned}$$

Make these substutions in (9):

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} = xu^{-4/3}$$

Now simplify:

$$\begin{aligned}
-\frac{1}{3}\frac{du}{dx} + u &= x \\
\frac{du}{dx} - 3u &= -3x
\end{aligned}$$

(Either of the last two equations is an acceptable final answer.)

8. Set

$$\begin{aligned}
u &= y - x + 5; \\
\frac{du}{dx} &= \frac{dy}{dx} - 1.
\end{aligned}$$

Make these substitutions in the given equation:

$$\begin{aligned}\frac{dy}{dx} &= 1 + e^{y-x+5} \\ \frac{dy}{dx} - 1 &= e^{y-x+5} \\ \frac{du}{dx} &= e^{u}\end{aligned}$$

Now separate:

$$\begin{aligned}e^{-u}\frac{du}{dx} &= 1 \\ e^{-u}du &= dx\end{aligned}$$

(Either of the last two equations is an acceptable final answer.)