

1. Rewrite in differential form:

$$\begin{aligned} 2x + (2y - 2) \frac{dy}{dx} &= -3 \\ 2x + 3 + (2y - 2) \frac{dy}{dx} &= 0 \\ (2x + 3) dx + (2y - 2) dy &= 0 \end{aligned}$$

This differential is **exact** by the exactness criterion:

$$\begin{aligned} \frac{\partial}{\partial y} (2x + 3) &= 0; \\ \frac{\partial}{\partial x} (2y - 2) &= 0. \end{aligned}$$

We want f such that $df(x, y) = (2x + 3) dx + (2y - 2) dy$, that is

$$\begin{aligned} f_x(x, y) = 2x + 3 & \quad \text{and} \quad f_y(x, y) = 2y - 2 \\ f(x, y) = 2x^2 + 3x + g(y) & \quad \quad \quad \vdots \\ f_y(x, y) = g'(y) & \quad \text{and} \quad f_y(x, y) = 2y - 2 \end{aligned} \tag{1}$$

Combine the last pair of equations:

$$\begin{aligned} g'(y) &= 2y - 2 \\ g(y) &= y^2 - 2y \end{aligned} \tag{2}$$

Substitute (2) into equation (1):

$$f(x, y) = 2x^2 + 3x + y^2 - 2y$$

The general solution is

$$2x^2 + 3x + y^2 - 2y = C$$

...which will be made explicit:

$$\begin{aligned} y^2 - 2y &= -2x^2 - 3x + C \\ y^2 - 2y + 1 &= -2x^2 - 3x + C \quad (C := C + 1) \\ (y - 1)^2 &= C - 2x^2 - 3x + C \\ y - 1 &= \pm \sqrt{-2x^2 - 3x + C} \\ y &= 1 \pm \sqrt{-2x^2 - 3x + C} \end{aligned}$$

2. Rewrite in differential form:

$$\begin{aligned}\frac{dy}{dx} &= \frac{5x+4y}{8y^3-4x} \\ (8y^3-4x)\frac{dy}{dx} &= 5x+4y \\ (-5x-4y) + (8y^3-4x)\frac{dy}{dx} &= 0 \\ (-5x-4y)dx + (8y^3-4x)dy &= 0\end{aligned}$$

This differential is **exact**:

$$\begin{aligned}\frac{\partial}{\partial y}(-5x-4y) &= -4; \\ \frac{\partial}{\partial x}(8y^3-4x) &= -4.\end{aligned}$$

We want f such that $df(x, y) = (2x+3)dx + (2y-2)dy$, that is

$$\begin{aligned}f_x(x, y) &= -5x-4y & \text{and} & & f_y(x, y) &= 8y^3-4x \\ & \vdots & & & f(x, y) &= 2y^4-4xy+g(x) \\ f_x(x, y) &= -5x-4y & \text{and} & & f_x(x, y) &= -4y+g'(x)\end{aligned}\quad (3)$$

Combine the last pair of equations:

$$\begin{aligned}-4y+g'(x) &= -5x-4y \\ g'(x) &= -5x \\ g(y) &= -\frac{5}{2}x^2\end{aligned}\quad (4)$$

Substitute (4) into (3):

$$f(x, y) = 2y^4 - 4xy - \frac{5}{2}x^2$$

The general solution is

$$2y^4 - 4xy - \frac{5}{2}x^2 = C$$

...which will be left implicit.

3. Rewrite in differential form:

$$\begin{aligned}e^x \sin y - 2y \sin x + (e^x \cos y + 2 \cos x)\frac{dy}{dx} &= 0 \\ (e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2 \cos x)dy &= 0\end{aligned}$$

This differential is **exact**:

$$\begin{aligned}\frac{\partial}{\partial y}(e^x \sin y + 2y \sin x) &= e^x \cos y + 2 \sin x; \\ \frac{\partial}{\partial x}(e^x \cos y - 2 \cos x) &= e^x \cos y + 2 \sin x.\end{aligned}$$

We want f such that $df(x, y) = (e^x \sin y - 2y \sin x) dx + (e^x \cos y + 2 \cos x) dy$, that is

$$\begin{aligned} f_x(x, y) &= e^x \sin y + 2y \sin x & \text{and} & & f_y(x, y) &= e^x \cos y + 2 \cos x \\ f(x, y) &= e^x \sin y - 2y \cos x + g(y) & & & & \vdots \\ f_y(x, y) &= e^x \cos y - 2 \cos x + g'(y) & \text{and} & & f_x(x, y) &= e^x \cos y + 2 \cos x \end{aligned} \quad (5)$$

Combine the last pair of equations:

$$\begin{aligned} e^x \cos y - 2 \cos x + g'(y) &= e^x \cos y + 2 \cos x \\ g'(y) &= 0 \\ g(y) &= 0 \end{aligned} \quad (6)$$

Substitute (6) into (5):

$$f(x, y) = e^x \sin y - 2y \cos x$$

The general solution is

$$e^x \sin y - 2y \cos x = C$$

...which will be left implicit.

4. Rewrite in differential form:

$$\begin{aligned} x \ln y + xy + (y \ln x + xy) \frac{dy}{dx} &= 0 \\ (x \ln y + xy) dx + (y \ln x + xy) dy &= 0 \end{aligned}$$

This differential **is not exact**:

$$\begin{aligned} \frac{\partial}{\partial y} (x \ln y + xy) dx &= \frac{x}{y} + x; \\ \frac{\partial}{\partial x} (y \ln x + xy) &= \frac{y}{x} + y. \end{aligned}$$

(These two are not equal, for example, when $x = 1$ and $y = 2$.)

5. Rewrite in differential form:

$$\begin{aligned} x - y^3 + y^2 \sin x + (-3xy^2 - 2y \cos x) \frac{dy}{dx} &= 0 \\ (x - y^3 + y^2 \sin x) dx + (-3xy^2 - 2y \cos x) dy &= 0 \end{aligned}$$

This differential **is exact**:

$$\begin{aligned} \frac{\partial}{\partial y} (x - y^3 + y^2 \sin x) &= -3y^2 + 2y \sin x; \\ \frac{\partial}{\partial x} (-3xy^2 - 2y \cos x) &= -3y^2 + 2y \sin x. \end{aligned}$$

We want f such that $df(x, y) = (x - y^3 + y^2 \sin x) dx + (-3xy^2 - 2y \cos x) dy$,
that is

$$\begin{aligned} f_x(x, y) &= x - y^3 + y^2 \sin x & \text{and} & & f_y(x, y) &= -3xy^2 - 2y \cos x \\ f(x, y) &= \frac{1}{2}x^2 - xy^3 - y^2 \cos x + g(y) & & & & \vdots \\ f_y(x, y) &= -3xy^2 - 2y \cos x + g'(y) & \text{and} & & f_y(x, y) &= -3xy^2 - 2y \cos x \end{aligned} \quad (7)$$

Combine the last two equations:

$$\begin{aligned} -3xy^2 - 2y \cos x + g'(y) &= -3xy^2 - 2y \cos x \\ g'(y) &= 0 \\ g(y) &= 0 \end{aligned} \quad (8)$$

Substitute (8) into (7):

$$f(x, y) = \frac{1}{2}x^2 - xy^3 - y^2 \cos x$$

The general solution is

$$\frac{1}{2}x^2 - xy^3 - y^2 \cos x = C$$

...which will be left implicit.

6. Set $u = \frac{y}{x}$. Then

$$\begin{aligned} y &= xu; \\ \frac{dy}{dx} &= \frac{d}{dx}(xu) \\ &= u + x \frac{du}{dx}. \end{aligned}$$

Make these substitutions in the given equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2 + 2y}{y^2} \\ u + x \frac{du}{dx} &= \frac{(xu)^2 + 2x(xu)}{(xu)^2} \end{aligned}$$

Now simplify and separate:

$$\begin{aligned}u + x \frac{du}{dx} &= \frac{x^2 u^2 + 2x^2 u}{x^2 u^2} \\ &= \frac{2 + u}{u} \\ x \frac{du}{dx} &= \frac{2 + u - u^2}{u} \\ \frac{du}{dx} &= \frac{2 + u - u^2}{xu} \\ \frac{u}{2 + u - u^2} \frac{du}{dx} &= \frac{1}{x} \\ \frac{u}{2 + u - u^2} du &= \frac{1}{x} dx\end{aligned}$$

(Either of the last two equations is an acceptable final answer.)

7. Rewrite in standard Bernoulli form:

$$\frac{dy}{dx} + y = xy^4 \quad (9)$$

Set $u = y^{-3}$. Then

$$\begin{aligned}y &= u^{-1/3}; \\ \frac{dy}{dx} &= -\frac{1}{3}u^{-4/3} \frac{du}{dx}.\end{aligned}$$

Make these substitutions in (9):

$$-\frac{1}{3}u^{-4/3} \frac{du}{dx} + u^{-1/3} = xu^{-4/3}$$

Now simplify:

$$\begin{aligned}-\frac{1}{3} \frac{du}{dx} + u &= x \\ \frac{du}{dx} - 3u &= -3x\end{aligned}$$

(Either of the last two equations is an acceptable final answer.)

8. Set

$$\begin{aligned}u &= y - x + 5; \\ \frac{du}{dx} &= \frac{dy}{dx} - 1.\end{aligned}$$

Make these substitutions in the given equation:

$$\begin{aligned}\frac{dy}{dx} &= 1 + e^{y-x+5} \\ \frac{dy}{dx} - 1 &= e^{y-x+5} \\ \frac{du}{dx} &= e^u\end{aligned}$$

Now separate:

$$\begin{aligned}e^{-u} \frac{du}{dx} &= 1 \\ e^{-u} du &= dx\end{aligned}$$

(Either of the last two equations is an acceptable final answer.)