1.) $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=0} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$

<u>for $f(x) = \sin(x)$:</u>

$f(0) = \sin(0) = 0$ AND $f^{(4k)}(0) = \sin(0) = 0$

$f'(0) = \cos(0) = 1$ $f^{(4k+1)}(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$ $f^{(4k+2)}(0) = -\sin(0) = 0$

$f'''(0) = -\cos(0) = -1$ $f^{(4k+3)}(0) = -\cos(0) = -1$

$$\sin(x) = \frac{1}{0!}(0)x^0 + \frac{1}{1!}(1)x^1 + \frac{1}{2!}(0)x^2 + \frac{1}{3!}(-1)x^3 + \frac{1}{4!}(0)x^4 + \frac{1}{5!}(1)x^5 + \ldots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{(2k+1)}$$

<u>for $f(x) = \cos(x)$:</u>

$f(0) = \cos(0) = 1$ AND $f^{(4k)}(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$ $f^{(4k+1)}(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$ $f^{(4k+2)}(0) = -\cos(0) = -1$

$f'''(0) = \sin(0) = 0$ $f^{(4k+3)}(0) = \sin(0) = 0$

$$\cos(x) = \frac{1}{0!}(1)x^0 + \frac{1}{1!}(0)x^1 + \frac{1}{2!}(-1)x^2 + \frac{1}{3!}(0)x^3 + \frac{1}{4!}(1)x^4 + \frac{1}{5!}(0)x^5 + \ldots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

2.) $\frac{d}{dx}(\sin(x)) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\right)$

$$= \sum_{k=0}^{\infty}\left(\frac{d}{dx}\left(\frac{(-1)^k}{(2k+1)!} x^{2k+1}\right)\right) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k)!(2k+1)}\left[\frac{d}{dx}\left(x^{2k+1}\right)\right]\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(2k+1)}\left[(2k+1)\, x^{2k}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos(x).$$

3.) $y'' - xy = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \; ; \; y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \; ; \; y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$n-2 = m+1$

$n = m+3$

$$\sum_{m=-3}^{\infty} (m+3)(m+2) c_{m+3} x^{m+1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\underset{(n=-3)}{0} + \underset{(n=-2)}{0} + \underset{(n=-1)}{2c_2} + \sum_{n=0}^{\infty} [(n+3)(n+2) c_{n+3} - c_n] x^{n+1} = 0$$

$2c_2 = 0 \Rightarrow c_2 = 0$ and $(n+3)(n+2) c_{n+3} - c_n = 0$

$$c_{n+3} = \frac{c_n}{(n+3)(n+2)}$$

$c_0 = c_0$	$c_1 = c_1$	$c_2 = 0$
$c_3 = \frac{c_0}{6} = \frac{c_0}{2\cdot3}$	$c_4 = \frac{c_1}{3\cdot4}$	$c_5 = 0$
$c_6 = \frac{c_3}{5\cdot6} = \frac{c_0}{2\cdot3\cdot5\cdot6}$	$c_7 = \frac{c_4}{6\cdot7} = \frac{c_1}{3\cdot4\cdot6\cdot7}$	$c_8 = 0$
$c_9 = \frac{c_6}{8\cdot9} = \frac{c_0}{2\cdot3\cdot5\cdot6\cdot8\cdot9}$	$c_{10} = \frac{c_7}{9\cdot10} = \frac{c_1}{3\cdot4\cdot6\cdot7\cdot9\cdot10}$	$c_{11} = 0$
$\vdots$	$\vdots$	$\vdots$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \underset{c_2=0}{c_2 x^2} + c_3 x^3 + c_4 x^4 + \ldots$$

$$= c_0 + c_1 x + \frac{c_0}{2\cdot3} x^3 + \frac{c_1}{3\cdot4} x^4 + \frac{c_0}{2\cdot3\cdot5\cdot6} x^6 + \frac{c_1}{3\cdot4\cdot6\cdot7} x^7 + \ldots$$

$$y(x) = c_0 \left[1 + \frac{1}{2\cdot3} x^3 + \frac{1}{2\cdot3\cdot5\cdot6} x^6 + \frac{1}{2\cdot3\cdot5\cdot6\cdot8\cdot9} x^9 + \ldots \right]$$

$$+ c_1 \left[x + \frac{1}{3\cdot4} x^4 + \frac{1}{3\cdot4\cdot6\cdot7} x^7 + \frac{1}{3\cdot4\cdot6\cdot7\cdot9\cdot10} x^{10} + \ldots \right]$$

And since x is analytic everywhere, the radius of convergence for this solution is $R = \infty$.

4.) $y'' + x^2 y' + xy = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \; ; \; y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \; ; \; y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + x^2 \sum_{n=0}^{\infty} n c_n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$n-2 = m+1$

$n = m+3$

$$\sum_{m=-3}^{\infty} (m+3)(m+2) c_{m+3} x^{m+1} + \sum_{n=0}^{\infty} [n c_n + c_n] x^{n+1} = 0$$

$$0 + 0 + 2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) c_{n+3} + (n+1) c_n] x^{n+1} = 0$$

$2c_2 = 0 \Rightarrow c_2 = 0$ and $(n+3)(n+2) c_{n+3} + (n+1) c_n = 0$

$$c_{n+3} = \frac{-(n+1) c_n}{(n+2)(n+3)}$$

$c_0 = c_0$	$c_1 = c_1$	$c_2 = 0$
$c_3 = \frac{-c_0}{2 \cdot 3}$	$c_4 = \frac{-2c_1}{3 \cdot 4}$	$c_5 = 0$
$c_6 = \frac{-4c_3}{5 \cdot 6} = \frac{4c_0}{2 \cdot 3 \cdot 5 \cdot 6}$	$c_7 = \frac{-5c_4}{6 \cdot 7} = \frac{2 \cdot 5 c_1}{3 \cdot 4 \cdot 6 \cdot 7}$	$c_8 = 0$
$c_9 = \frac{-7c_6}{8 \cdot 9} = \frac{-(4 \cdot 7) c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$	$c_{10} = \frac{-8c_7}{9 \cdot 10} = \frac{-(2 \cdot 5 \cdot 8) c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}$	$c_{11} = 0$
$\vdots$	$\vdots$	$\vdots$

$$y(x) = c_0 + c_1 x - \frac{c_0}{2 \cdot 3} x^3 - \frac{2c_1}{3 \cdot 4} x^4 + \frac{4c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{2 \cdot 5 c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \ldots$$

$$y(x) = c_0 \left[1 - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 4}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1 \cdot 4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + - \ldots \right]$$

$$+ c_1 \left[x - \frac{2}{3 \cdot 4} x^4 + \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + - \ldots \right]$$

And since x^2 and x are analytic everywhere, the radius of convergence for this solution is $R = \infty$

5.) $(x-1)y'' + y' = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \quad ; \quad y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \quad ; \quad y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$(x-1)\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^{n-1} = 0$$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^{n-1} = 0$$

$n-2 = m-1$

$n = m+1$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1} - \sum_{m=-1}^{\infty} (m+1)(m)c_{m+1} x^{m-1} + \sum_{n=0}^{\infty} n c_n x^{n-1} = 0$$

$$\sum_{n=0}^{\infty} [n(n-1)c_n - (n+1)(n)c_{n+1} + n c_n] x^{n-1} = 0$$

$$-(n+1)(n)c_{n+1} + (n(n-1)+n)c_n = 0$$

$$c_{n+1} = \frac{n^2 c_n}{n(n+1)} = \frac{n c_n}{n+1} \quad \text{when} \quad n \neq 0$$

$c_0 = c_0$

$c_1 = c_1$

$c_2 = \frac{c_1}{2}$

$c_3 = \frac{2c_2}{3} = \frac{c_1}{3}$

$c_4 = \frac{3c_3}{4} = \frac{c_1}{4}$

$\vdots$

$c_n = \frac{c_1}{n}$

$$y(x) = c_0 + c_1 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \ldots$$

$$= c_0 + c_1 \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) x^n$$

And this solution has a radius of convergence of $R = 1$ since $\frac{1}{(x-1)}$ has a singularity at $x = 1$.

6.) $(x-1)y'' - xy' + y = 0 \qquad y(0) = -2, \quad y'(0) = 6$

$$y = \sum_{n=0}^{\infty} c_n x^n \; ; \; y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \; ; \; y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$$

$$(x-1)\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$n-1 = m \qquad n-2 = k$

$n = m+1 \qquad n = k+2$

$$\sum_{m=-1}^{\infty}(m+1)(m)c_{m+1}x^m - \sum_{k=-2}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{n=0}^{\infty}(-n+1)c_n x^n = 0$$

$$0 + \sum_{n=0}^{\infty}(n+1)(n)c_{n+1}x^n - \left[0 + 0 + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n\right] + \sum_{n=0}^{\infty}(1-n)c_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n)c_{n+1} - (n+2)(n+1)c_{n+2} + (1-n)c_n\right]x^n = 0$$

$$(n+1)n c_{n+1} - (n+2)(n+1)c_{n+2} + (1-n)c_n = 0$$

$$c_{n+2} = \frac{n(n+1)c_{n+1} + (1-n)c_n}{(n+2)(n+1)}$$

$c_0 = c_0$

$c_2 = \frac{c_0}{2}$

$c_3 = \frac{2c_2}{6} = \frac{c_0}{6}$

$c_4 = \frac{6c_3 - c_2}{12} \quad \frac{c_0}{24}$

$\vdots$

$c_k = \frac{c_0}{k!}$ for $k \geq 2$

$c_1 = c_1$

$$y(x) = c_0 + c_1 x + \sum_{k=2}^{\infty} c_0 \frac{1}{k!} x^k$$

$$= c_0 + c_0 x - c_0 x + c_1 x + c_0 \sum_{k=2}^{\infty} \frac{x^k}{k!}$$

$$= \left[c_0 + c_0 x + \sum_{k=2}^{\infty} \frac{x^k}{k!}\right] - c_0 x + c_1 x$$

$$y(x) = c_0 e^x - c_0 x + c_1 x$$

$y(0) = c_0 e^0 - c_0(0) + c_1(0) = -2$

$= c_0 = -2$

$y'(x) = c_0 e^x - c_0 + c_1$

$y'(0) = c_0 - c_0 + c_1 = 6$

$c_1 = 6$

$y(x) = -2e^x + 2x + 6x$

$= -2e^x + 8x.$