1: *"Find the power series expansions for* $\sin x$ *and* $\cos x$ *about* $x = 0$ *using the Taylor expansion formula."*

The derivatives of $\sin x$ at $x = 0$ are

- $\sin 0 = 0$;
- $\left(\frac{d}{dx}\sin x\right)_{x=0} = \cos 0 = 1$;
- $\left(\frac{d^2}{dx^2}\sin x\right)_{x=0} = -\sin 0 = 0$;
- $\left(\frac{d^3}{dx^3}\sin x\right)_{x=0} = -\cos 0 = -1$;
- $\left(\frac{d^4}{dx^4}\sin x\right)_{x=0} = \sin 0 = 0$, etc.

So

$$\begin{aligned} \sin x &= \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \\ &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}x^{2k+1}. \end{aligned}$$

Likewise, the derivatives of $\cos x$ at $x = 0$ are

- $\cos 0 = 1$;
- $\left(\frac{d}{dx}\cos x\right)_{x=0} = -\sin 0 = 0$;
- $\left(\frac{d^2}{dx^2}\cos x\right)_{x=0} = -\cos 0 = -1$;
- $\left(\frac{d^3}{dx^3}\cos x\right)_{x=0} = \sin 0 = 0$;
- $\left(\frac{d^4}{dx^4}\cos x\right)_{x=0} = \cos 0 = 1$, etc.

So

$$\begin{aligned} \cos x &= \frac{1}{0!} + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \cdots \\ &= \frac{1}{0!} - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}x^{2k}. \end{aligned}$$

2: *"Use the power series expansions from Problem 1 to show that* $\frac{d}{dx}\sin x = \cos x$.*"*

$$\begin{aligned} \frac{d}{dx}\sin x &= \frac{d}{dx}\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(2k+1)\,x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}x^{2k} \\ &= \cos x. \end{aligned}$$

3: $y'' - xy = 0$

Recall that

$$\begin{aligned} y &= \sum_{k=0}^{\infty} c_k x^k; \\ y' &= \sum_{k=1}^{\infty} k c_k x^{k-1}; \\ y'' &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}. \end{aligned}$$

Substitute these power series into $y'' - xy$.

$$\begin{aligned} y'' - xy &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - x \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0}^{\infty} c_k x^{k+1} \\ &= \sum_{k=-1}^{\infty} (k+3)(k+2) c_{k+3} x^{k+1} - \sum_{k=0}^{\infty} c_k x^{k+1} \\ &= 2 \cdot 1 \cdot c_2 + \sum_{k=0}^{\infty} (k+3)(k+2) c_{k+3} x^{k+1} - \sum_{k=0}^{\infty} c_k x^{k+1} \\ &= 2c_2 + \sum_{k=0}^{\infty} \left[(k+3)(k+2) c_{k+3} - c_k\right] x^{k+1} \\ &= 2c_2 + \sum_{k=3}^{\infty} \left[k(k-1) c_k - c_{k-3}\right] x^{k-2} \end{aligned}$$

The equation becomes

$$2c_2 + \sum_{k=3}^{\infty} \left[k(k-1) c_k - c_{k-3}\right] x^{k-2} = 0$$

which means

- $c_2 = 0$;
- $c_k = \frac{1}{k(k-1)} c_{k-3}$, for all $k \geq 3$.

The general solution is

$$\begin{aligned} y &= c_0 + c_1 x + \frac{1}{6} c_0 x^3 + \frac{1}{12} c_1 x^4 + \frac{1}{6 \cdot 30} c_0 x^6 + \frac{1}{12 \cdot 42} c_1 x^7 + \frac{1}{6 \cdot 30 \cdot 72} c_0 x^9 + \frac{1}{12 \cdot 42 \cdot 90} c_1 x^{10} + \cdots \\ &= c_0 \left(1 + \frac{1}{6} x^3 + \frac{1}{12 \cdot 42} x^6 + \frac{1}{6 \cdot 30 \cdot 72} x^9 \cdots \right) + c_1 \left(x + \frac{1}{12} x^4 + \frac{1}{12 \cdot 42} x^7 + \frac{1}{12 \cdot 42 \cdot 90} x^{10} \cdots \right). \end{aligned}$$

Two specific solutions are

$$\begin{aligned} y_1 &= 1 + \frac{1}{6} x^3 + \frac{1}{12 \cdot 42} x^6 + \frac{1}{6 \cdot 30 \cdot 72} x^9 + \cdots \\ y_2 &= x + \frac{1}{12} x^4 + \frac{1}{12 \cdot 42} x^7 + \frac{1}{12 \cdot 42 \cdot 90} x^{10} + \cdots \end{aligned}$$

...both of which have radius of convergence ∞.

4: $y'' + x^2y' + xy = 0$

Rewrite y, y', and y'' as power series.

$$\begin{aligned}
y'' + x^2y' + xy &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + x^2 \sum_{k=1}^{\infty} kc_k x^{k-1} + x\sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=1}^{\infty} kc_k x^{k+1} + \sum_{k=0}^{\infty} c_k x^{k+1} \\
&= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=4}^{\infty} (k-3)c_{k-3} x^{k-2} + \sum_{k=3}^{\infty} c_{k-3} x^{k-2} \\
&= 2c_2 + 6c_3x + \sum_{k=4}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=4}^{\infty} (k-3)c_{k-3} x^{k-2} + c_0 + \sum_{k=4}^{\infty} c_{k-3} x^{k-2} \\
&= (2c_2 + c_0) + 6c_3x + \sum_{k=4}^{\infty} \left[k(k-2)c_k + (k-3)c_{k-3} + c_{k-3}\right] x^{k-2} \\
&= (2c_2 + c_0) + 6c_3x + \sum_{k=4}^{\infty} \left[k(k-2)c_k + (k-2)c_{k-3}\right] x^{k-2}
\end{aligned}$$

The equation becomes

$$(2c_2 + c_0) + 6c_3x + \sum_{k=4}^{\infty} \left[k(k-2)c_k + (k-2)c_{k-3}\right] x^{k-2} = 0$$

which means

- $c_2 = -\frac{1}{2}c_0$;
- $c_3 = 0$;
- $c_k = -\frac{1}{k}c_{k-3}$, for all $k \geq 4$.

The general solution is

$$\begin{aligned}
y &= c_0 + c_1x - \frac{1}{2}c_0x^2 - \frac{1}{4}c_1x^4 + \frac{1}{2\cdot5}c_0x^5 + \frac{1}{4\cdot7}c_1x^7 - \frac{1}{2\cdot5\cdot8}c_0x^8 - \frac{1}{4\cdot7\cdot10}c_1x^{10} + \cdots \\
&= c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{2\cdot5}x^5 - \frac{1}{2\cdot5\cdot8}x^8 + \cdots\right) + c_1\left(x - \frac{1}{4}x^4 + \frac{1}{4\cdot7}x^7 - \frac{1}{4\cdot7\cdot10}x^{10} + \cdots\right)
\end{aligned}$$

Two specific solutions are

$$\begin{aligned}
y_1 &= 1 - \frac{1}{2}x^2 + \frac{1}{2\cdot5}x^5 - \frac{1}{2\cdot5\cdot8}x^8 + \cdots \\
y_2 &= x - \frac{1}{4}x^4 + \frac{1}{4\cdot7}x^7 - \frac{1}{4\cdot7\cdot10}x^{10} + \cdots
\end{aligned}$$

...both of which have radius of convergence ∞.

5: $(x-1)y'' + y' = 0$

Rewrite y' and y'' as power series.

$$\begin{aligned}
(x-1)\,y''+y' &= (x-1)\sum_{k=2}^{\infty}k\,(k-1)\,c_k x^{k-2}+\sum_{k=1}^{\infty}kc_k x^{k-1} \\
&= \sum_{k=2}^{\infty}k\,(k-1)\,c_k x^{k-1}-\sum_{k=2}^{\infty}k\,(k-1)\,c_k x^{k-2}+\sum_{k=1}^{\infty}kc_k x^{k-1} \\
&= \sum_{k=3}^{\infty}(k-1)\,(k-2)\,c_{k-1}x^{k-2}-\sum_{k=2}^{\infty}k\,(k-1)\,c_k x^{k-2}+\sum_{k=2}^{\infty}(k-1)\,c_{k-1}x^{k-2} \\
&= \sum_{k=3}^{\infty}(k-1)\,(k-2)\,c_{k-1}x^{k-2}-2c_2-\sum_{k=3}^{\infty}k\,(k-1)\,c_k x^{k-2}+c_1+\sum_{k=3}^{\infty}(k-1)\,c_{k-1}x^{k-2} \\
&= (-2c_2+c_1)+\sum_{k=3}^{\infty}\left[(k-1)\,(k-2)\,c_{k-1}-k\,(k-1)\,c_k+(k-1)\,c_{k-1}\right]x^{k-2} \\
&= -\,(2c_2-c_1)-\sum_{k=3}^{\infty}\left[k\,(k-1)\,c_k-(k-1)\,(k-1)\,c_{k-1}\right]x^{k-2}
\end{aligned}$$

The equation becomes

$$-\,(2c_2-c_1)-\sum_{k=3}^{\infty}\left[k\,(k-1)\,c_k-(k-1)\,(k-1)\,c_{k-1}\right]x^{k-2}=0$$

which means

- $c_2=\frac{1}{2}c_1$;
- $c_k=\frac{k-1}{k}c_{k-1}$, for all $k\geq 3$.

The general solution is

$$\begin{aligned}
y &= c_0+c_1x+\frac{1}{2}c_1x^2+\frac{1}{3}c_1x^3+\frac{1}{4}c_1x^4+\cdots \\
&= c_0+c_1\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\right) \\
&= c_0+c_1\ln(1-x)\,.
\end{aligned}$$

Two specific solutions are

$$\begin{aligned}
y_1 &= 1; \\
y_2 &= \ln(1-x)\,.
\end{aligned}$$

The former solution has radius of convergence ∞; the latter has radius of convergence 1.

6: $(x-1)\,y''-xy'+y=0$; $y(0)=-2$; $y'(0)=6$

Rewrite using power series, noting that $c_0 = y(0) = -2$ and $c_1 = y'(0) = 6$:

$$\begin{aligned}
(x-1)y'' - xy' + y &= (x-1)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - x\sum_{k=1}^{\infty} kc_k x^{k-1} + \sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-1} - \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{k=3}^{\infty} (k-1)(k-2)c_{k-1} x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=3}^{\infty} (k-2)c_{k-2} x^{k-2} + \sum_{k=2}^{\infty} c_{k-2} x^{k-2} \\
&= \sum_{k=3}^{\infty} (k-1)(k-2)c_{k-1} x^{k-2} - 2c_2 - \sum_{k=3}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=3}^{\infty} (k-2)c_{k-2} x^{k-2} + c_0 + \sum_{k=3}^{\infty} c_{k-2} x^{k-2} \\
&= -2c_2 - 2 + \sum_{k=3}^{\infty} \left[(k-1)(k-2)c_{k-1} - k(k-1)c_k - (k-2)c_{k-2} + c_{k-2}\right] x^{k-2} \\
&= -2(c_2+1) - \sum_{k=3}^{\infty} \left[k(k-1)c_k - (k-1)(k-2)c_{k-1} + (k-3)c_{k-2}\right] x^{k-2}.
\end{aligned}$$

The equation becomes

$$-2(c_2+1) - \sum_{k=3}^{\infty} \left[k(k-1)c_k - (k-1)(k-2)c_{k-1} + (k-3)c_{k-2}\right] x^{k-2} = 0$$

which means

- $c_2 = -1$;
- $c_k = \frac{k-2}{k} c_{k-1} - \frac{k-3}{k(k-1)} c_{k-2}$, for all $k \geq 3$.

In particular,

- $c_3 = \frac{1}{3} c_2 - \frac{0}{6} c_1 = -\frac{1}{3}$;
- $c_4 = \frac{2}{4} c_3 - \frac{1}{12} c_2 = -\frac{1}{6} + \frac{1}{12} = -\frac{1}{12}$;
- $c_4 = \frac{3}{5} c_4 - \frac{2}{20} c_3 = -\frac{1}{20} + \frac{1}{30} = -\frac{1}{60}$, etc.

This means

$$y = -2 + 6x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5 - \cdots.$$