1.) (a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \mathcal{L}^{-1}\left\{\left.\frac{1}{s^n}\right|_{s\to s-a}\right\}$$

$$= e^{at}\,\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}\,e^{at}\,\mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\} = \frac{1}{(n-1)!}\,e^{at}\,t^{n-1}.$$

(b.) $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-as}\,\frac{1}{s}\right\} = \mathcal{U}(t-a)\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Big|_{t\to t-a}$$

$$= \mathcal{U}(t-a)\,(1)\Big|_{t\to t-a} = \mathcal{U}(t-a).$$

(c.) $\mathcal{L}^{-1}\left\{e^{-as}\,\frac{1}{(s-b)^2+k^2}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-as}\,\frac{1}{(s-b)^2+k^2}\right\} = \mathcal{U}(t-a)\,\mathcal{L}^{-1}\left\{\frac{1}{(s-b)^2+k^2}\right\}\Big|_{t\to t-a}$$

$$= \mathcal{U}(t-a)\,\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2+k^2}\right|_{s\to s-b}\right\}\Big|_{t\to t-a}$$

$$= \frac{1}{k}\,\mathcal{U}(t-a)\,\mathcal{L}^{-1}\left\{\left.\frac{k}{s^2+k^2}\right|_{s\to s-b}\right\}\Big|_{t\to t-a}$$

$$= \frac{1}{k}\,\mathcal{U}(t-a)\left[e^{bt}\,\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}\right]_{t\to t-a}$$

$$= \frac{1}{k}\,\mathcal{U}(t-a)\,e^{b(t-a)}\,\sin(k(t-a)).$$

(d.) $\mathcal{L}\{t^2\sin(kt)\}$

$$\mathcal{L}\{t^2\sin(kt)\} = (-1)^2\,\frac{d^2}{ds^2}\,\mathcal{L}\{\sin(kt)\} = \frac{d^2}{ds^2}\left(\frac{k}{s^2+k^2}\right) = \frac{d^2}{ds^2}\left(k(s^2+k^2)^{-1}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(k(s^2+k^2)^{-1}\right)\right) = \frac{d}{ds}\left(-k(s^2+k^2)^{-2}(2s)\right) = \frac{d}{ds}\left(\frac{-2sk}{(s^2+k^2)^2}\right)$$

$$= \frac{(s^2+k^2)^2(-2k) - (-2sk)(2(s^2+k^2)(2s))}{((s^2+k^2)^2)^2} = \frac{-2k(s^2+k^2)+8s^2k}{(s^2+k^2)^3} = \frac{6s^2k-2k^3}{(s^2+k^2)^3}$$

2.) $\mathcal{L}^{-1}\left\{ \frac{1}{s^2(s+1)^2} \right\}$

$\mathcal{L}^{-1}\left\{ \frac{1}{s^2(s+1)^2} \right\} = \mathcal{L}^{-1}\left\{ \left(\frac{1}{s^2}\right)\left(\frac{1}{(s+1)^2}\right) \right\}$ let $F(s) = \frac{1}{(s+1)^2} \Rightarrow f(t) = te^{-t}$

$G(s) = \frac{1}{s^2} \Rightarrow g(t) = t$

$= f(t) * g(t)$

$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

$= \int_0^t \tau e^{-\tau}(t-\tau)\, d\tau = \int_0^t (t\tau e^{-\tau} - \tau^2 e^{-\tau})\, d\tau$

$= \left(t\int \tau e^{-\tau}\, d\tau - \int \tau^2 e^{-\tau}\, d\tau\right)\Big|_0^t$ $u = \tau^2$ $v = -e^{-\tau}$

$du = 2\tau\, d\tau$ $dv = e^{-\tau} d\tau$

$= \left(t\int \tau e^{-\tau}\, d\tau - \left[-\tau^2 e^{-\tau} + 2\int \tau e^{-\tau}\, d\tau\right]\right)\Big|_0^t$

$= \left(\tau^2 e^{-\tau} + (t-2)\int \tau e^{-\tau}\, d\tau\right)\Big|_0^t$ $u = \tau$ $v = -e^{-\tau}$

$du = d\tau$ $dv = e^{-\tau} d\tau$

$= \left(\tau^2 e^{-\tau} + (t-2)\left[-\tau e^{-\tau} + \int e^{-\tau}\, d\tau\right]\right)\Big|_0^t$

$= \left(\tau^2 e^{-\tau} + (t-2)\left[-\tau e^{-\tau} - e^{-\tau}\right]\right)\Big|_0^t$

$= t^2 e^{-t} + (t-2)(-t e^{-\tau} - e^{-t} - (0-1))$

$= t^2 e^{-t} + (-t^2 e^{-t} - te^{-t} + t + 2te^{-t} + 2e^{-t} - 2)$

$= te^{-t} + t + 2e^{-t} - 2$

Using partial fractions:

$\frac{1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$

$1 = s(s+1)^2 A + B(s+1)^2 + C(s^2(s+1)) + D(s^2)$

$1 = (s^3 + 2s^2 + s)A + (s^2 + 2s + 1)B + (s^3 + s^2)C + s^2 D$

s^3: $0 = A + C$ $\quad$ s: $0 = A + 2B$

s^2: $0 = 2A + B + C + D$ $\quad$ const: $1 = B$ $\Rightarrow A = -2,\ C = 2,\ D = 1$

$\mathcal{L}\left\{ \frac{1}{s^2(s+1)^2} \right\} = \mathcal{L}\left\{ \frac{-2}{s} \right\} + \mathcal{L}\left\{ \frac{1}{s^2} \right\} + \mathcal{L}\left\{ \frac{2}{s+1} \right\} + \mathcal{L}\left\{ \frac{1}{(s+1)^2} \right\}$

$= -2 + t + 2e^{-t} + te^{-t}$

✓

3.) $\mathcal{L}\{\cosh(kt)\}$

$$\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{kt} - e^{-kt}\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-k)} + \frac{1}{(s+k)}\right] = \frac{1}{2}\left[\frac{s+k+(s-k)}{(s-k)(s+k)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-k^2}\right] = \frac{s}{s^2-k^2}.$$

4.) $\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

Since $\mathcal{L}\{f(t)\} = F(s)$, we have:

- $\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}F(s)$

Taking the inverse Laplace transform of both sides yields:

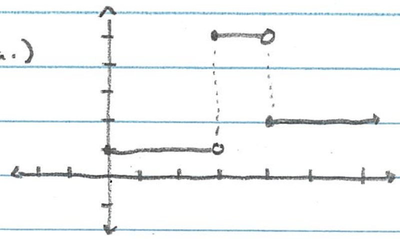
- $\mathcal{L}^{-1}\{\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\}\} = \mathcal{L}^{-1}\{e^{-as}F(s)\}$

$$\mathcal{U}(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

$$\mathcal{U}(t-a)\, f(t)\Big|_{t\to t-a} = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

$$\mathcal{U}(t-a)\,\mathcal{L}^{-1}\{F(s)\}\Big|_{t\to t-a} = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

5.) (a.)

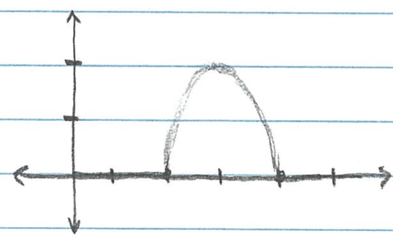


$$f(t) = 1 + 4\mathcal{U}(t-3) - 3\mathcal{U}(t-4)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + 4\mathcal{L}\{\mathcal{U}(t-3)\} - 3\mathcal{L}\{\mathcal{U}(t-4)\}$$

$$F(s) = \frac{1}{s} + 4\left(\frac{e^{-3s}}{s}\right) - 3\left(\frac{e^{-4s}}{s}\right)$$

(b.)



$$g(t) = \begin{cases} 0 & 0 \le t < 2 \\ 2(1-(t-3)^2) & 2 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

$$g(t) = 0 + (2(1-(t-3)^2))\,\mathcal{U}(t-2) - (2(1-(t-3)^2))\,\mathcal{U}(t-4)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\mathcal{U}(t-2)(2(1-(t-3)^2))\} - \mathcal{L}\{\mathcal{U}(t-4)(2(1-(t-3)^2))\}$$

$$G(s) = 2e^{-2s}\,\mathcal{L}\{(1-(t-3)^2)\big|_{t\to t+2}\} - 2e^{-4s}\,\mathcal{L}\{(1-(t-3)^2)\big|_{t\to t+4}\}$$

$$G(s) = 2e^{-2s}\,\mathcal{L}\{(1-(t-1)^2)\} - 2e^{-4s}\,\mathcal{L}\{(1-(t+1)^2)\}$$

$$= 2e^{-2s}\,\mathcal{L}\{-t^2+2t\} - 2e^{-4s}\,\mathcal{L}\{-t^2-2t\}$$

$$= 2e^{-2s}\left(\frac{-2}{s^3} + \frac{2}{s^2}\right) - 2e^{-4s}\left(\frac{-2}{s^3} - \frac{2}{s^2}\right)$$

$$= 4e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^3}\right) + 4e^{-4s}\left(\frac{1}{s^2} + \frac{1}{s^3}\right)$$

6.) $y'' + y' + y = 1 + e^{-t} \qquad y(0) = 3, \quad y'(0) = -5$

$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{e^{-t}\}$

$s^2 Y(s) - s\,y(0) - y'(0) + sY(s) - y(0) + Y(s) = \frac{1}{s} + \frac{1}{s+1}$

$(s^2 + s + 1)Y(s) - 3s + 2 = \frac{1}{s} + \frac{1}{s+1}$

$$Y(s) = \frac{1}{s(s^2+s+1)} + \frac{1}{(s+1)(s^2+s+1)} + \frac{3s}{s^2+s+1} - \frac{2}{s^2+s+1}$$

Note: $\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$

$1 = A(s^2+s+1) + s(Bs+C)$

(s^2 terms): $0 = A + B$

(s terms): $0 = A + C$

(constants): $1 = A \quad \Rightarrow B = -1, C = -1$

and, $\frac{1}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$

$1 = A(s^2+s+1) + (s+1)(Bs+C)$

(s^2 terms): $0 = A + B$

(s terms): $0 = A + B + C$

(constants): $1 = A + C \quad \Rightarrow B = -1, A = 1, C = 0.$

$$Y(s) = \left[\frac{1}{s} - \frac{s}{s^2+s+1} - \frac{1}{s^2+s+1}\right] + \left[\frac{1}{s+1} - \frac{s}{s^2+s+1}\right] + \frac{3s}{s^2+s+1} - \frac{2}{s^2+s+1}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s+1} + \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{3}{4}} - \frac{\frac{7}{2}}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\} - \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}$$

$$y(t) = 1 + e^{-t} + \mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+\frac{3}{4}}\right|_{s\to s+\frac{1}{2}}\right\} - \frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left\{\left.\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}\right|_{s\to s+\frac{1}{2}}\right\}$$

$$y(t) = 1 + e^{-t} + e^{-\frac{1}{2}t}\cos\left(\tfrac{\sqrt{3}}{2}t\right) - \tfrac{7}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\tfrac{\sqrt{3}}{2}t\right).$$

7.) $y'' + 2y' + y = 3\delta(t-1) \qquad y(0) = y'(0) = 0$

$$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = 3\mathcal{L}\{\delta(t-1)\}$$

$$(s+1)^2 Y(s) = 3e^{-s}$$

$$Y(s) = 3\left(e^{-s}\frac{1}{(s+1)^2}\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{3e^{-s}\frac{1}{(s+1)^2}\right\}$$

$$y(t) = 3\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2}\right\}$$

$$y(t) = 3u(t-1)\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\Big|_{t \to t-1}$$

$$y(t) = 3u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s \to s+1}\right\}\right]_{t \to t-1}$$

$$y(t) = 3u(t-1)\left[e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right]_{t \to t-1}$$

$$y(t) = 3u(t-1)\left[e^{-t}t\right]\Big|_{t \to t-1}$$

$$y(t) = 3u(t-1)\left[e^{-(t-1)}(t-1)\right]$$

8.) $y'' + 4y = f(t) = \begin{cases} \cos(t) & 0 \le t < \pi/2 \\ 0 & t \ge \pi/2 \end{cases}$ $\qquad y'(0) = y(0) = 0$

$$y'' + 4y = \cos(t) - \cos(t)\,\mathcal{U}(t - \pi/2)$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\cos(t)\} - \mathcal{L}\{\cos(t)\,\mathcal{U}(t-\pi/2)\}$$

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{s}{s^2+1} - e^{-\pi/2\, s}\,\mathcal{L}\{\cos(t+\pi/2)\}$$

$\cos(t + \pi/2) = -\sin(t)$

$$(s^2+4)\,Y(s) = \frac{s}{s^2+1} + e^{-\pi/2\, s}\,\mathcal{L}\{\sin(t)\}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2+4)} + e^{-\pi/2\, s}\left(\frac{1}{(s^2+1)(s^2+4)}\right)$$

Note: $\dfrac{s}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$$s = (s^2+4)(As+B) + (s^2+1)(Cs+D)$$

$$s = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D$$

s^3: $0 = A + C$

s^2: $0 = B + D$ $\quad \Rightarrow A = 1/3,\ C = -1/3$

s: $1 = 4A + C$ $\quad \Rightarrow B = 0,\ D = 0$

$0 = 4B + D$

and: $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{\bar{A}s+\bar{B}}{s^2+1} + \dfrac{\bar{C}s+\bar{D}}{s^2+4}$

$$1 = \bar{A}s^3 + \bar{B}s^2 + 4\bar{A}s + 4\bar{B} + \bar{C}s^3 + \bar{D}s^2 + \bar{C}s + \bar{D}$$ (similar to above)

s^3: $0 = \bar{A} + \bar{C}$

s^2: $0 = \bar{B} + \bar{D}$ $\quad \Rightarrow \bar{A} = 0,\ \bar{C} = 0$

s: $0 = 4\bar{A} + \bar{C}$ $\quad \Rightarrow \bar{B} = 1/3,\ \bar{C} = -1/3$

: $1 = 4\bar{B} + \bar{D}$

$$Y(s) = \left[\frac{1}{3}\left(\frac{s}{s^2+1}\right) - \frac{1}{3}\left(\frac{s}{s^2+4}\right)\right] + e^{-\pi/2\, s}\left[\frac{1}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{1}{s^2+4}\right)\right]$$

NEXT $\Rightarrow$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{1/3\left(\frac{s}{s^2+1}\right)\} - \mathcal{L}^{-1}\{1/3\left(\frac{s}{s^2+4}\right)\} + \mathcal{L}^{-1}\{1/3 e^{-\pi/2 s}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]\}$$

$$y(t) = 1/3\cos(t) - 1/3\cos(2t) + 1/3\,\mathcal{U}(t-\pi/2)\,\mathcal{L}^{-1}\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\}\Big|_{t\to t-\pi/2}$$

$$y(t) = 1/3\cos(t) - 1/3\cos(2t) + 1/3\,\mathcal{U}(t-\pi/2)\left[\sin(t) - \frac{1}{2}\sin(2t)\right]\Big|_{t\to t-\pi/2}$$

$$y(t) = 1/3\cos(t) - 1/3\cos(2t) + 1/3\,\mathcal{U}(t-\pi/2)\left[\sin(t-\pi/2) - \frac{1}{2}\sin(2t-\pi)\right]$$

OR (since $\sin(t-\pi/2) = -\cos(t)$, $\sin(2t-\pi) = -\sin(2t)$)

$$y(t) = 1/3\cos(t) - 1/3\cos(2t) + 1/3\,\mathcal{U}(t-\pi/2)\left[-\cos(t) + 1/2\sin(2t)\right].$$