[Name]
[Section #]
Homework #7
MATH 527

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(1) *"Use Laplace transforms to solve for y, where $y'' + y = \sin(t)$; $y(0) = 1$; $y'(0) = 2$."*

Apply the Laplace transform.

$$\begin{aligned} \mathscr{L}\{y''+y\} &= \mathscr{L}\{\sin(t)\} \\ \mathscr{L}\{y''\}+\mathscr{L}\{y\} &= \mathscr{L}\{\sin(t)\} \\ \left(s^2Y - sy(0) - y'(0)\right) + Y &= \frac{1}{s^2+1} \end{aligned}$$

Use the given initial conditions and solve for Y.

$$\begin{aligned} \left(s^2Y - s\cdot 1 - 2\right) + Y &= \frac{1}{s^2+1}; \\ \left(s^2+1\right)Y - (s+2) &= \frac{1}{s^2+1}; \\ \left(s^2+1\right)Y &= \frac{1}{s^2+1} + s + 2; \\ Y &= \frac{1}{\left(s^2+1\right)^2} + \frac{s}{s^2+1} + \frac{2}{s^2+1} \end{aligned}$$

Invert the Laplace transform.

$$\begin{aligned} y &= \mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2} + \frac{s}{s^2+1} + \frac{2}{s^2+1}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{\sin(t) - t\cos(t)}{2} + \cos(t) + 2\sin(t) \\ &= \frac{5}{2}\sin(t) + \cos(t) - \frac{1}{2}t\cos(t). \end{aligned}$$

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(2) *"Use Laplace transforms to solve for y, where $y'' - 4y' + 4y = t^2e^{2t}$; $y(0) = 0$; $y'(0) = 0$."*

Apply the Laplace transform.

$$\begin{aligned} \mathscr{L}\{y''-4y'+4y\} &= \mathscr{L}\{t^2e^{2t}\} \\ \mathscr{L}\{y''\} - 4\mathscr{L}\{y'\} + 4\mathscr{L}\{y\} &= \mathscr{L}\{t^2\}_{s\mapsto s-2} \\ \left(s^2Y - sy(0) - y'(0)\right) - 4\left(sY - y(0)\right) + 4Y &= \left[\frac{2!}{s^3}\right]_{s\mapsto s-2} \end{aligned}$$

Use the given initial conditions and solve for Y.

$$\begin{aligned} \left(s^2-4s+4\right)Y &= \frac{2}{(s-2)^3} \\ (s-2)^2\,Y &= \frac{2}{(s-2)^3} \\ Y &= \frac{2}{(s-2)^5} \end{aligned}$$

Invert the Laplace transform.

$$\begin{aligned} y &= \mathscr{L}^{-1}\left\{\frac{2}{(s-2)^5}\right\} \\ &= \frac{1}{3\cdot 4}\mathscr{L}^{-1}\left\{\frac{4!}{(s-2)^5}\right\} \\ &= \frac{1}{12}t^4 e^{2t} \end{aligned}$$

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(3) *"Evaluate $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} 2 & t<3 \\ -2 & 3\le t \end{cases}$."*

Rewrite in terms of $\mathscr{U}$.

$$f(t) = 2 - 4\mathscr{U}(t-3)$$

Apply the Laplace transform.

$$\begin{aligned} F(s) &= \mathscr{L}\{2-4\mathscr{U}(t-3)\} \\ &= 2\mathscr{L}\{1\} - 4\mathscr{L}\{\mathscr{U}(t-3)\} \\ &= 2\cdot\mathscr{L}\{1\} - 4e^{-3s}\mathscr{L}\{1\} \\ &= \left(2-4e^{-3s}\right)\mathscr{L}\{1\} \\ &= \frac{2-4e^{-3s}}{s} \end{aligned}$$

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(4) *"Evaluate $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} 0 & 0\le t<\frac{3\pi}{2} \\ \sin(t) & \frac{3\pi}{2}\le t \end{cases}$."*

Rewrite in terms of $\mathscr{U}$.

$$\begin{aligned} f(t) &= \sin(t)\,\mathscr{U}\left(t-\frac{3\pi}{2}\right) \\ &= -\cos\left(t-\frac{3\pi}{2}\right)\mathscr{U}\left(t-\frac{3\pi}{2}\right) \end{aligned}$$

Apply the Laplace transform.

$$\begin{aligned} F(s) &= \mathscr{L}\left\{-\cos\left(t-\frac{3\pi}{2}\right)\mathscr{U}\left(t-\frac{3\pi}{2}\right)\right\} \\ &= -e^{-\frac{3\pi}{2}s}\mathscr{L}\{\cos(t)\} \\ &= -e^{-\frac{3\pi}{2}s}\cdot\frac{s}{s^2+1} \\ &= -\frac{se^{-\frac{3\pi}{2}s}}{s^2+1} \end{aligned}$$

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(5) *"Use Laplace transforms to solve for y, where $y''+4y = \begin{cases} 1 & 0\le t<1 \\ 0 & 1\le t \end{cases}$; $y(0)=0$; $y'(0)=-1$."*

Rewrite in terms of $\mathscr{U}$.

$$y''+4y = 1-\mathscr{U}(t-1).$$

Apply the Laplace transform.

$$\begin{aligned}
\mathscr{L}\{y''+4y\} &= \mathscr{L}\{1-\mathscr{U}(t-1)\}\\
\mathscr{L}\{y''\}+4\mathscr{L}\{y\} &= \mathscr{L}\{1\}-\mathscr{L}\{\mathscr{U}(t-1)\}\\
\left(s^2Y-sy(0)-y'(0)\right)+4Y &= \frac{1}{s}-\frac{e^{-s}}{s}
\end{aligned}$$

Use the given initial values and solve for Y.

$$\begin{aligned}
\left(s^2Y+1\right)+4Y &= \frac{1}{s}-\frac{e^{-s}}{s}\\
\left(s^2+4\right)Y+1 &= \frac{1}{s}-\frac{e^{-s}}{s}\\
\left(s^2+4\right)Y &= \frac{1}{s}-\frac{e^{-s}}{s}-1\\
Y &= \frac{1}{s\left(s^2+4\right)}-\frac{e^{-s}}{s\left(s^2+4\right)}-\frac{1}{s^2+4}
\end{aligned}$$

Decompose $\frac{1}{s(s^2+4)}$ into partial fractions.

$$\begin{aligned}
\frac{A}{s}+\frac{Bs+C}{s^2+4} &= \frac{1}{s\left(s^2+4\right)}\\
A\left(s^2+4\right)+(Bs+C)s &= 1\\
(A+B)s^2+Cs+4A &= 1
\end{aligned}$$

Break this into three equations.

$$\begin{array}{ccc}
A+B=0 & C=0 & 4A=1\\
\frac{1}{4}+B=0 & \longleftarrow & A=\frac{1}{4}\\
B=-\frac{1}{4} & &
\end{array}$$

Therefore

$$\frac{1}{s\left(s^2+4\right)}=\frac{1}{4}\left(\frac{1}{s}\right)-\frac{1}{4}\left(\frac{s}{s^2+4}\right).$$

Substitute into the expression for Y, then invert the Laplace transform.

$$\begin{aligned}
Y &= \frac{1}{4}\left(\frac{1}{s}\right)-\frac{1}{4}\left(\frac{s}{s^2+4}\right)-\frac{1}{4}e^{-s}\left(\frac{1}{s}\right)+\frac{1}{4}e^{-s}\left(\frac{s}{s^2+4}\right)-\frac{1}{s^2+4}\\
&= \frac{1}{4}\left(\frac{1}{s}\right)-\frac{1}{s^2+4}-\frac{1}{4}\left(\frac{s}{s^2+4}\right)-\frac{1}{4}e^{-s}\left(\frac{1}{s}\right)+\frac{1}{4}e^{-s}\left(\frac{s}{s^2+4}\right)\\
y &= \mathscr{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s}\right)-\frac{1}{s^2+4}-\frac{1}{4}\left(\frac{s}{s^2+4}\right)-\frac{1}{4}e^{-s}\left(\frac{1}{s}\right)+\frac{1}{4}e^{-s}\left(\frac{s}{s^2+4}\right)\right\}\\
&= \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}-\frac{1}{4}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{1}{4}\mathscr{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s}\right\}+\frac{1}{4}\mathscr{L}^{-1}\left\{e^{-s}\cdot\frac{s}{s^2+4}\right\}\\
&= \frac{1}{4}-\frac{1}{2}\sin(2t)-\frac{1}{4}\cos(2t)-\frac{1}{4}\mathscr{U}(t-1)+\frac{1}{4}\cos(2(t-1))\mathscr{U}(t-1)\\
&= \frac{1-2\sin(2t)-\cos(2t)}{4}+\frac{(\cos(2(t-1))-1)}{4}\mathscr{U}(t-1)
\end{aligned}$$

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(6) *"Use Laplace transforms to solve for y, where* $y''+2y'+y=\begin{cases}0 & 0\le t<3\\ 2(t-3) & 3\le t\end{cases}$; $y(0)=2;\ y'(0)=1.$*"*

Rewrite in terms of $\mathscr{U}$.

$$y''+2y'+y=2(t-3)\mathscr{U}(t-3)$$

Take the Laplace transform.

$$\begin{aligned}
\mathscr{L}\left\{y'' + 2y' + y\right\} &= \mathscr{L}\left\{2(t-3)\mathscr{U}(t-3)\right\} \\
\mathscr{L}\left\{y''\right\} + 2\mathscr{L}\left\{y'\right\} + \mathscr{L}\left\{y\right\} &= 2e^{-3s}\mathscr{L}\left\{t\right\} \\
\left(s^2Y - sy(0) - y'(0)\right) + 2\left(sY - y(0)\right) + Y &= 2e^{-3s}\cdot\frac{1}{s^2} \\
\left(s^2 + 2s + 1\right)Y - (s+2)\,y(0) - y'(0) &= 2e^{-3s}\cdot\frac{1}{s^2}
\end{aligned}$$

Use the given initial values and solve for Y.

$$\begin{aligned}
\left(s^2 + 2s + 1\right)Y - (s+2)\cdot 2 - 1 &= 2e^{-3s}\cdot\cdot\frac{1}{s^2} \\
(s+1)^2 Y &= -2e^{-3s}\cdot\frac{1}{s^2} + 2(s+2) + 1 \\
Y &= 2e^{-3s}\cdot\frac{1}{s^2(s+1)^2} + 2\cdot\frac{s+2}{(s+1)^2} + \frac{1}{(s+1)^2} \\
&= 2e^{-3s}\cdot\frac{1}{s^2(s+1)^2} + 2\cdot\frac{1}{s+1} + 3\cdot\frac{1}{(s+1)^2}
\end{aligned}$$

Decompose $\frac{1}{s^2(s+1)^2}$ into partial fractions.

$$\begin{aligned}
\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} &= \frac{1}{s^2(s+1)^2} \\
As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2 &= 1 \\
(A+C)s^3 + (2A+B+C+D)s^2 + (A+2B)s + B &= 1
\end{aligned}$$

This gives the system of equations

$$A + C = 0;\ 2A + B + C + D = 0;\ A + 2B = 0;\ B = 1.$$

Eliminating B we get

$$A + C = 0;\ 2A + C + D = -1;\ A = -2.$$

Eliminating A we get

$$C = 2;\ C + D = 3.$$

Eliminating C we get

$$D = 1.$$

So in fact

$$Y = 2e^{-3s}\left(-2\cdot\frac{1}{s} + \frac{1}{s^2} + 2\cdot\frac{1}{s+1} + \frac{1}{(s+1)^2}\right) + 2\cdot\frac{1}{s+1} + 3\cdot\frac{1}{(s+1)^2}.$$

Invert the Laplace transform.

$$\begin{aligned}
y &= \mathscr{L}^{-1}\left\{2e^{-3s}\left(-2\cdot\frac{1}{s} + \frac{1}{s^2} + 2\cdot\frac{1}{s+1} + \frac{1}{(s+1)^2}\right) + 2\cdot\frac{1}{s+1} + 3\cdot\frac{1}{(s+1)^2}\right\} \\
&= 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 4\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} + 2\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} + 4\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s+1}\right\} + 2\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\} \\
&= 2e^{-t} + 3e^{-t}t - 4\mathscr{U}(t-3) + 2(t-3)\mathscr{U}(t-3) + 4e^{-(t-3)}\mathscr{U}(t-3) + 2e^{-(t-3)}(t-3)\mathscr{U}(t-3) \\
&= (3t-2)e^{-t} + 2\left(t - 5 + (2t-1)e^{-(t-3)}\right)\mathscr{U}(t-3)
\end{aligned}$$

(7) *"Use Laplace transforms to solve for y, where $y'' + y = \sin(t) + \delta(t-\pi)$; $y(0) = 0$; $y'(0) = 0$."*

Apply the Laplace transform.

$$\begin{aligned}
\mathscr{L}\{y'' + y\} &= \mathscr{L}\{\sin(t) + \delta(t-\pi)\} \\
\mathscr{L}\{y''\} + \mathscr{L}\{y\} &= \mathscr{L}\{\sin(t)\} + \mathscr{L}\{\delta(t-\pi)\} \\
\left(s^2 Y - s y(0) - y'(0)\right) + Y &= \frac{1}{s^2+1} + e^{-\pi s}
\end{aligned}$$

Use the given initial values and solve for Y.

$$\begin{aligned}
s^2 Y + Y &= \frac{1}{s^2+1} + e^{-\pi s} \\
\left(s^2+1\right) Y &= \frac{1}{s^2+1} + e^{-\pi s} \\
Y &= \frac{1}{\left(s^2+1\right)^2} + e^{-\pi s} \cdot \frac{1}{s^2+1}
\end{aligned}$$

Invert the Laplace transform.

$$\begin{aligned}
y &= \mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} + \mathscr{L}^{-1}\left\{e^{-\pi s} \cdot \frac{1}{s^2+1}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{t \mapsto t-\pi} \mathscr{U}(t-\pi) \\
&= \frac{\sin(t) - t\cos(t)}{2} + \sin(t-\pi)\,\mathscr{U}(t-\pi) \\
&= \frac{\sin(t) - t\cos(t)}{2} - \sin(t)\,\mathscr{U}(t-\pi)
\end{aligned}$$