1.) $\mathcal{L}\{e^{\alpha t}\} = \int_0^\infty e^{-st} e^{\alpha t}\, dt$

$$= \lim_{b\to\infty} \int_0^b e^{(\alpha - s)t}\, dt$$

$$= \lim_{b\to\infty} \left[\frac{1}{(\alpha - s)} e^{(\alpha - s)t} \Big|_0^b \right]$$

$$= 0 - \frac{1}{(\alpha - s)} = \frac{1}{s-a}$$

2.) $\mathcal{L}\{\sin(kt)\} = \int_0^\infty e^{-st} \sin(kt)\, dt$ $\quad u = \sin(kt) \quad v = -\frac{1}{s} e^{-st}$

$du = k\cos(kt)dt \quad dv = e^{-st}dt$

$$= \lim_{b\to\infty} \left[\left(-\frac{1}{s} e^{-st} \sin(kt) - \int \frac{-k}{s} \cos(kt) e^{-st}\, dt \right) \Big|_0^b \right]$$

$$= \left[0 + \frac{1}{s} e^0 \sin(k(0)) + \frac{k}{s} \int_0^\infty \cos(kt) e^{-st}\, dt \right]$$

$$= \frac{k}{s} \left(\mathcal{L}\{\cos(kt)\} \right)$$

$\mathcal{L}\{\cos(kt)\} = \int_0^\infty e^{-st} \cos(kt)\, dt$ $\quad u = \cos(kt) \quad v = -\frac{1}{s} e^{-st}$

$du = -k\sin(kt)dt \quad dv = e^{-st}dt$

$$= \lim_{b\to\infty} \left[\left(-\frac{1}{s} e^{-st} \cos(kt) - \int \frac{k}{s} e^{-st} \sin(kt)\, dt \right) \Big|_0^b \right]$$

$$= \left[0 + \frac{1}{s} - \frac{k}{s} \int_0^\infty e^{-st} \sin(kt)\, dt \right]$$

$$= \frac{1}{s} - \frac{k}{s} \mathcal{L}\{\sin(kt)\}$$

Let $A = \mathcal{L}\{\sin(kt)\}$, $B = \mathcal{L}\{\cos(kt)\}$. *then*

$$A = \frac{k}{s} B$$

$$B = \frac{1}{s} - \frac{k}{s} A \implies B = \frac{1}{s} - \frac{k}{s}\left(\frac{k}{s} B\right)$$

$$B = \frac{1}{s} - \frac{k^2}{s^2} B$$

$$B\left(1 + \frac{k^2}{s^2}\right) = \frac{1}{s} \Rightarrow B = \frac{1}{s}\left(\frac{1}{(1 + k^2/s^2)}\right) = \frac{s}{s^2+k^2}$$

$$A = \frac{k}{s^2+k^2}$$

So $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

and $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$

3.) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt$

$= \lim_{b\to\infty} \left(\frac{-1}{s} e^{-st}\right)\Big|_0^b$

$= [0 - (-\frac{1}{s})] = \frac{1}{s}$

$\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t)\, dt$ $\quad u = t \quad v = \frac{-1}{s} e^{-st}$

$du = dt \quad dv = e^{-st}\, dt$

$= \lim_{b\to\infty} \left(te^{-st} - \int -\frac{1}{s}(e^{-st})\, dt\right)\Big|_0^b$

$= [0 - 0] + \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}(t^n)\, dt$ $\quad u = t^n \quad v = -\frac{1}{s} e^{-st}$

$du = nt^{n-1}\, dt \quad dv = e^{-st}\, dt$

$= \lim_{b\to\infty} \left(-\frac{1}{s}e^{-st}(t^n) - \int -\frac{1}{s}(e^{-st})(nt^{n-1})\, dt\right)\Big|_0^b$

$= [0 - 0] + \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$

Reapplying the formula yields : $\mathcal{L}\{t^n\} = \frac{n}{s}\left(\frac{n-1}{s}\mathcal{L}\{t^{n-2}\}\right)$

$\vdots$

$\mathcal{L}\{t^n\} = \frac{n!}{s^n}\mathcal{L}\{1\}$

$= \frac{n!}{s^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}}$

So $\mathcal{L}\{1\} = \frac{1}{s}$

$(n=1)$ $\mathcal{L}\{t\} = \frac{1}{s^2}$ $\quad$ and $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$(n=2)$ $\mathcal{L}\{t^2\} = \frac{2}{s^3}$

$(n=3)$ $\mathcal{L}\{t^3\} = \frac{6}{s^4}$

4.) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{s^2+2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= \cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$$

$$= \cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)$$

5) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}$

Note: $\frac{1}{s^2+s-20} = \frac{1}{(s+5)(s-4)} = \frac{A}{s+5} + \frac{B}{s-4}$

$$\Rightarrow \quad 1 = A(s-4) + B(s+5)$$

@ $s = 4$: $1 = B(9)$, $B = 1/9$

@ $s = -5$: $1 = A(-9)$, $A = -1/9$

So $\frac{1}{s^2+s-20} = -\frac{1}{9}\left(\frac{1}{s+5}\right) + \frac{1}{9}\left(\frac{1}{s-4}\right)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{9}\left(\frac{1}{s+5}\right) + \frac{1}{9}\left(\frac{1}{s-4}\right)\right\}$$

$$= -\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= -\frac{1}{9}e^{-5t} + \frac{1}{9}e^{4t}.$$

6.) $\mathcal{L}^{-1}\left\{\dfrac{2s-4}{(s^2+s)(s^2+1)}\right\}$

Note: $\dfrac{2s-4}{s(s+1)(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{Cs+D}{s^2+1}$

-2

$$\Rightarrow 2s-4 = A(s+1)(s^2+1) + B(s)(s^2+1) + (Cs+D)(s+1)(s)$$

$$2s-4 = A(s^3+s^2+s+1) + B(s^3+s) + (Cs^3+Cs^2+Ds^2+Ds)$$

(s^3 terms): $0 = A+B+C$ $\qquad\qquad$ $0 = -4+B+C \Rightarrow 4-C = B$

(s^2 terms): $0 = A+C+D$ $\qquad \Rightarrow \left\{\begin{array}{l} 0 = -4+C+D \\ 2 = -4+B+D \end{array}\right\}$

(s terms): $2 = A+B+D$

(const. terms): $-4 = A$

$$\Downarrow$$

$$4 = C+D$$

$$2 = -C+D$$

$$6 = 2D \Rightarrow D=3,\ C=1,\ B=3$$

So $\dfrac{2s-4}{(s^2+s)(s^2+1)} = \dfrac{-4}{s} + \dfrac{3}{s+1} + \dfrac{s+3}{s^2+1}$

$$= -\frac{4}{s} + \frac{3}{s+1} + \frac{s}{s^2+1} + \frac{3}{s^2+1}.$$

$$\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{-\frac{4}{s} + \frac{3}{s+1} + \frac{s}{s^2+1} + \frac{3}{s^2+1}\right\}$$

$$= -4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -4(1) + 3(e^{-t}) + \cos(t) + 3\sin(t)$$

$$= -4 + 3e^{-t} + \cos(t) + 3\sin(t).$$

7.) $\frac{dy}{dt} - y = 1, \quad y(0) = 0$

$y' - y = 1$ · Take Laplace transform of both sides

$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\}$

$[sY(s) - y(0)] - [Y(s)] = \frac{1}{s}$

$sY(s) - Y(s) = \frac{1}{s}$

$Y(s)(s-1) = \frac{1}{s}$

$Y(s) = \frac{1}{s(s-1)}$

Using partial fractions: $\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$

$1 = A(s-1) + B(s)$

@ $s=1$ @ $s=0$

$1 = B$ $1 = -A \Rightarrow A = -1.$

$Y(s) = -\frac{1}{s} + \frac{1}{s-1}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left(-\frac{1}{s} + \frac{1}{s-1}\right)$

$y(t) = -\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$

$y(t) = -1 + e^{t}.$