[Name]
[Section #]
Homework #5
Math 527

1

The equation to solve is

$$\left(D^2+4\right)y=\sin(x)\,. \tag{1.1}$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2+4\right)y_* &= 0\\ (D-2i)(D-2i)\,y_* &= 0\\ y_* &= C_1\cos(2x)+C_2\sin(2x)\end{aligned} \tag{1.2}$$

Now make the **judicious guess**

$$y=\alpha\cos(x)+\beta\sin(x)\,. \tag{1.3}$$

Then

$$\begin{aligned}\left(D^2+4\right)y &= \left(-\alpha\cos(x)-\beta\sin(x)\right)+4\left(\alpha\cos(x)+\beta\sin(x)\right)\\ &= 3\alpha\cos(x)+3\beta\sin(x)\,;\end{aligned}$$

substituting this expression into (1.1) gives

$$3\alpha\cos(x)+3\beta\sin(x)=\sin(x)\,,$$

which implies

$$\alpha=0 \text{ and } \beta=\frac{1}{3}.$$

Substitute these values into (1.3).

$$y=\frac{1}{3}\sin(x)$$

The sum of this solution with (1.2) is the general solution.

$$y=\frac{1}{3}\sin(x)+C_1\cos(2x)+C_2\sin(2x)$$

2

$$\pi(F|data) = \pi(F) \cdot \frac{dPr\left(\langle \mathbf{x}, \mathbf{y} \rangle | F\right)}{\int_{f \in F}} \int_{f \in \mathbf{F}} dPr$$

The equation to solve is

$$\left(D^2 + 4\right) y = \sin(2x). \tag{2.1}$$

We already have the homogeneous solution in (1.2).

The **judicious guess** $y = \alpha \cos(2x) + \beta \sin(2x)$ won't work, since it's already a homogeneous solution. So instead use

$$y = x\left(\alpha \cos(2x) + \beta \sin(2x)\right). \tag{2.2}$$

Then, by the product rule,

$$\begin{aligned} D^2 y &= 2\left(-2\alpha \sin(2x) + 2\beta \cos(2x)\right) + \left(-4\alpha \cos(2x) - 4\beta \sin(2x)\right) \\ &= 4\beta \cos(2x) - 4\alpha \sin(2x) - 4x\left(\alpha \cos(2x) + \beta \cos(2x)\right), \end{aligned}$$

so

$$\left(D^2 + 4\right) y = 4\beta \cos(2x) - 4\alpha \sin(2x).$$

Substituting this expression into (2.1) gives

$$4\beta \cos(2x) - 4\alpha \sin(2x) = \sin(2x).$$

Solve for α and β.

$$\begin{array}{ccc} 4\beta = 0 & \text{and} & -4\alpha = 1 \\ \beta = 0 & \text{and} & \alpha = -\frac{1}{4} \end{array}$$

Substitute these values into (2.2).

$$y = -\frac{1}{4} x \cos(2x)$$

The sum of this solution with (1.2) is the general solution.

$$y = -\frac{1}{4} x \cos(2x) + C_1 \cos(2x) + C_2 \sin(2x)$$

3

The equation to solve is

$$\left(D^2-4D+10\right)y=e^{-t}\sin(t). \tag{3.1}$$

Solve the associated homogeneous equation.

$$\left(D^2-4D+10\right)y_* = 0$$

Solve the characteristic equation.

$$\begin{aligned} \lambda^2-4\lambda+10 &= 0 \\ \lambda &= \frac{4\pm\sqrt{16-40}}{2} \\ &= 2\pm i\sqrt{6} \end{aligned}$$

These are two nonreal roots. So,

$$y_* = e^{2t}\left(C_1\cos\left(\sqrt{6}t\right)+C_2\sin\left(\sqrt{6}t\right)\right). \tag{3.2}$$

Now, make the **judicious guess**

$$y=e^{-t}\left(\alpha\cos(t)+\beta\sin(t)\right). \tag{3.3}$$

For convenience, substitute $s=\alpha\cos(t)+\beta\sin(t)$. Then

$$\begin{aligned} y &= e^{-t}s; \\ Dy &= e^{-t}\left(s'-s\right); \\ D^2y &= e^{-t}\left(s''-2s'+s\right). \end{aligned}$$

Combining these gives

$$\begin{aligned} \left(D^2-4D+10\right)y &= e^{-t}\left(\left(s''-2s'+s\right)-4\left(s'-s\right)+10s\right) \\ &= e^{-t}\left(s''-6s'+15s\right). \end{aligned}$$

Note that

$$\begin{aligned} 15s &= 15\alpha\cos(t) &+& \quad 15\beta\sin(t); \\ -6s' &= -6\beta\cos(t) &+& \quad 6\alpha\sin(t); \\ s'' &= -\alpha\cos(t) &+& \quad -\beta\sin(t); \end{aligned}$$

and therefore

$$\left(D^2-4D+10\right)y=e^{-t}\left(\left(14\alpha-6\beta\right)\cos(t)+\left(6\alpha+14\beta\right)\sin(t)\right).$$

Substitute this expression into (3.1).

$$e^{-t}\left(\left(14\alpha - 6\beta\right)\cos(t) + \left(6\alpha + 14\beta\right)\sin(t)\right) = e^{-t}\sin(t)$$

Solve for α and β.

$$\begin{array}{rclcrcl}
14\alpha - 6\beta & = & 0 & \text{and} & 6\alpha + 14\beta & = & 1 \\
\alpha & = & \frac{6}{14}\beta & \text{and} & 6\left(\frac{6}{14}\beta\right) + 14\beta & = & 1 \\
\vdots & \vdots & \vdots & \vdots & \frac{116}{7}\beta & = & 1 \\
\alpha & = & \frac{6}{14}\cdot\frac{7}{116} & \text{and} & \beta & = & \frac{7}{116} \\
 & = & \frac{3}{116} & & & &
\end{array}$$

Substitute these values into (3.3).

$$y = e^{-t}\left(\frac{3}{116}\cos(t) + \frac{7}{116}\sin(t)\right)$$

The sum of this solution with (3.2) is the general solution.

$$y = e^{-t}\left(\frac{3}{116}\cos(t) + \frac{7}{116}\sin(t)\right) + e^{2t}\left(C_1\cos\left(\sqrt{6}t\right) + C_2\sin\left(\sqrt{6}t\right)\right)$$

4

The equation to solve is

$$\left(D^3 - 2D^2 - 4D + 8\right) y = 6xe^{2x}. \tag{4.1}$$

Solve the associated homogeneous equation.

$$\begin{aligned}
\left(D^3 - 2D^2 - 4D + 8\right) y_* &= 0 \\
(D-2)^2 (D+2)\, y_* &= 0 \\
y_* &= (C_0 + C_1 x)\, e^{2x} + C_2 e^{-2x}
\end{aligned} \tag{4.2}$$

The obvious **judicious guess**, $y = e^{2x}(\alpha x + \beta)$, won't work since it's already a homogeneous solution. So instead use

$$y = e^{2x}\left(\alpha x^3 + \beta x^2\right). \tag{4.3}$$

For convenience, substitute $s = \alpha x^3 + \beta x^2$. Then

$$\begin{aligned}
y &= e^{2x} s; \\
Dy &= e^{2x}\left(s' + 2s\right); \\
D^2 y &= e^{2x}\left(s'' + 4s' + 4s\right); \\
D^3 y &= e^{2x}\left(s''' + 6s'' + 12s' + 8s\right).
\end{aligned}$$

Combining these gives

$$\begin{aligned}
\left(D^3 - 2D^2 - 4D + 8\right) y &= e^{2x}\left(\left(s''' + 6s'' + 12s' + 8s\right) - 2\left(s'' + 4s' + 4s\right) - 4\left(s' + 2s\right) + 8s\right) \\
&= \left(s''' - 2s''\right) e^{2x}.
\end{aligned}$$

Note that

$$\begin{aligned}
s'' &= 6\alpha x + 2\beta; \\
s''' &= 6\alpha;
\end{aligned}$$

and therefore

$$\begin{aligned}
\left(D^3 - 2D^2 - 4D + 8\right) y &= e^{2x}\left(6\alpha + 12\alpha x + 4\beta\right) \\
&= 12\alpha x e^{2x} + \left(6\alpha + 4\beta\right) e^{2x}.
\end{aligned}$$

Substitute this expression into (4.1).

$$12\alpha x e^{2x} + \left(6\alpha + 4\beta\right) e^{2x} = 6xe^{2x}$$

Solve for α and β.

$$\begin{array}{ccc} 12\alpha = 6 & \text{and} & 6\alpha + 4\beta = 0 \\ \alpha = \frac{1}{2} & \text{and} & 3 + 4\beta = 0 \\ & & \beta = -\frac{3}{4} \end{array}$$

Substitute these values into (4.3).

$$y = e^{2x}\left(\frac{1}{2}x^3 - \frac{3}{4}x^2\right)$$

The sum of this solution with (4.1) is the general solution.

$$y = e^{2x}\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + C_1 x + C_0\right)$$

5

The equation to solve is

$$\left(D^2+2D+1\right)y=e^{-t}\ln(t).$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2+2D+1\right)y_* &= 0\\ (D+1)^2 y_* &= 0\\ y_* &= C_1e^{-t}+C_2te^{-t}\end{aligned}$$

Now, use **variation of parameters**. The general solution will be

$$y=u_1e^{-t}+u_2te^{-t}$$

with u_1', u_2' such that

$$\begin{aligned}e^{-t}u_1'+te^{-t}u_2' &= 0;\\ -e^{-t}u_1'+(1-t)e^{-t}u_2' &= e^{-t}\ln(t).\end{aligned}$$

Divide both equations through by e^{-t}:

$$u_1'+tu_2' = 0; \tag{5.1}$$

$$-u_1'+(1-t)u_2' = \ln(t). \tag{5.2}$$

To isolate u_2, add (5.1) and (5.2).

$$\begin{aligned}u_2' &= \ln(t)\\ u_2 &= \int\ln(t)\\ &= t(\ln(t)-1)+C_2\end{aligned}$$

To get u_1, substitute u_2' into (5.1).

$$\begin{aligned}u_1'+t\ln(t) &= 0\\ u_1' &= -t\ln(t)\\ u_1 &= -\int t\ln(t)\\ &= -t^2\left(\frac{1}{2}\ln(t)-\frac{1}{4}\right)+C_1.\end{aligned}$$

Therefore the general solution is

$$y = \left(-t^2\left(\frac{1}{2}\ln(t) - \frac{1}{4}\right) + C_1\right)e^{-t} + \left(t\left(\ln(t) - 1\right) + C_2\right)te^{-t},$$

that is,

$$y = \frac{1}{2}t^2e^{-t}\ln(t) - \frac{3}{4}t^2e^{-t} + C_1e^{-t} + C_2te^{-t}.$$

6

The equation to solve is

$$\left(D^2-2D+2\right)y=e^x\sec(x).$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2-2D+2\right)y_* &= 0\\ (D-(1-i))(D-(1+i))\,y_* &= 0\\ y_* &= C_1e^x\cos(x)+C_2e^x\sin(x)\end{aligned}$$

Now, use **variation of parameters**. The general solution will be

$$y=u_1e^x\cos(x)+u_2e^x\sin(x),$$

where

$$\begin{aligned}u_1' &= \frac{-y_1f(x)}{y_1y_2'-y_2y_1'}\\ &= \frac{-[e^x\sin(x)][e^x\sec(x)]}{[e^x\cos(x)]\frac{d}{dx}[e^x\sin(x)]-[e^x\sin(x)]\frac{d}{dx}[e^x\cos(x)]}\\ &= -\frac{e^{2x}\tan(x)}{e^{2x}\cos(x)(\sin(x)+\cos(x))-e^{2x}\sin(x)(\cos(x)-\sin(x))}\\ &= -\frac{\tan(x)}{\sin(x)^2+\cos(x)^2}\\ &= -\tan(x);\\ u_1 &= \int-\tan(x)\,dx\\ &= -\ln(\cos(x))+C_1\end{aligned}$$

and

$$\begin{aligned}u_2' &= \frac{y_2f(x)}{y_1y_2'-y_2y_1'}\\ &= \frac{[e^x\cos(x)][e^x\sec(x)]}{[e^x\cos(x)]\frac{d}{dx}[e^x\sin(x)]-[e^x\sin(x)]\frac{d}{dx}[e^x\cos(x)]}\\ &= \frac{e^{2x}}{e^{2x}}\\ &= 1;\\ u_2 &= \int dx\\ &= x+C_2.\end{aligned}$$

Therefore the general solution is

$$y = (\ln(\cos(x)) + C_1)\, e^x \cos(x) + (x + C_2)\, e^x \sin(x),$$

that is,

$$y = e^x (\cos(x) \ln(\cos(x)) + x \sin(x) + C_1 \cos(x) + C_2 \sin(x)).$$

7

The equation to solve is equivalent to

$$\left(D^2+\frac{k}{m}\right)y \quad = \quad \frac{F}{m}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

Make the substitutions

$$w=\sqrt{\frac{k}{m}};\; a=\frac{F}{m} \tag{7.1}$$

to simplify to

$$\left(D^2+w^2\right)y=a\cdot\sin(wt). \tag{7.2}$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2+w^2\right)y_* &= 0\\ (D-iw)(D+iw)y_* &= 0\\ y_* &= C_1\cos(wt)+C_2\sin(wt)\end{aligned} \tag{7.3}$$

Now, make the judicious guess

$$y=t\left(\alpha\cos(wt)+\beta\sin(wt)\right). \tag{7.4}$$

Then

$$\begin{aligned}D^2y &= 2\left(-w\alpha\sin(wt)+w\beta\cos(wt)\right)+t\left(-w^2\alpha\cos(wt)-w^2\beta\sin(wt)\right)\\ &= 2w\beta\cos(wt)-2w\alpha\cos(wt)-w^2t\left(\alpha\cos(wt)+\beta\sin(wt)\right),\end{aligned}$$

so

$$\left(D^2+w^2\right)y=2w\beta\cos(wt)-2w\alpha\sin(wt).$$

Substitute this expression into (7.2).

$$2w\beta\cdot\cos(wt)-2w\alpha\cdot\sin(wt)=a\cdot\sin(2t)$$

Solve for α and β.

$$\alpha=-\frac{a}{2w}\text{ and }\beta=0$$

Substitute these values into (7.4).

$$y=-\frac{a}{2w}t\cos(wt) \tag{7.5}$$

The general solution is the sum of (7.5) and (7.3).

$$\begin{aligned} y &= -\frac{a}{2w} t\cos(wt) + C_1\cos(wt) + C_2\sin(wt) \\ y' &= -\frac{a}{2w}(\cos(wt) - wt\sin(wt)) - wC_2\sin(wt) + wC_1\sin(wt) \end{aligned}$$

Now, use the initial values $y(0) = 0$, $y'(0) = 0$.

$$\begin{aligned} 0\cos(0) + C_1\cos(0) + C_2\sin(0) &= 0 \\ \frac{a}{2w}(\cos(0) - 0\sin(0)) - wC_1\sin(0) + wC_2\cos(0) &= 0 \end{aligned}$$

The former equation reduces to

$$C_1 = 0;$$

the latter to

$$\begin{aligned} \frac{a}{2w} - wC_2 &= 0 \\ C_2 &= \frac{a}{2w^2}. \end{aligned}$$

Substitute these values back into (7.2).

$$\begin{aligned} y &= \frac{a}{2w^2}\sin(wt) - \frac{a}{2w} t\cos(wt) \\ &= \frac{a}{2w^2}(\sin(wt) - wt\cos(wt)) \end{aligned}$$

Reverse the substitutions from (7.1) for the specific solution.

$$y = \frac{F}{2k}\left(\sin\left(\sqrt{\frac{k}{m}}t\right) - \sqrt{\frac{k}{m}}t\cos\left(\sqrt{\frac{k}{m}}t\right)\right).$$

A plot of this function is on the following page.

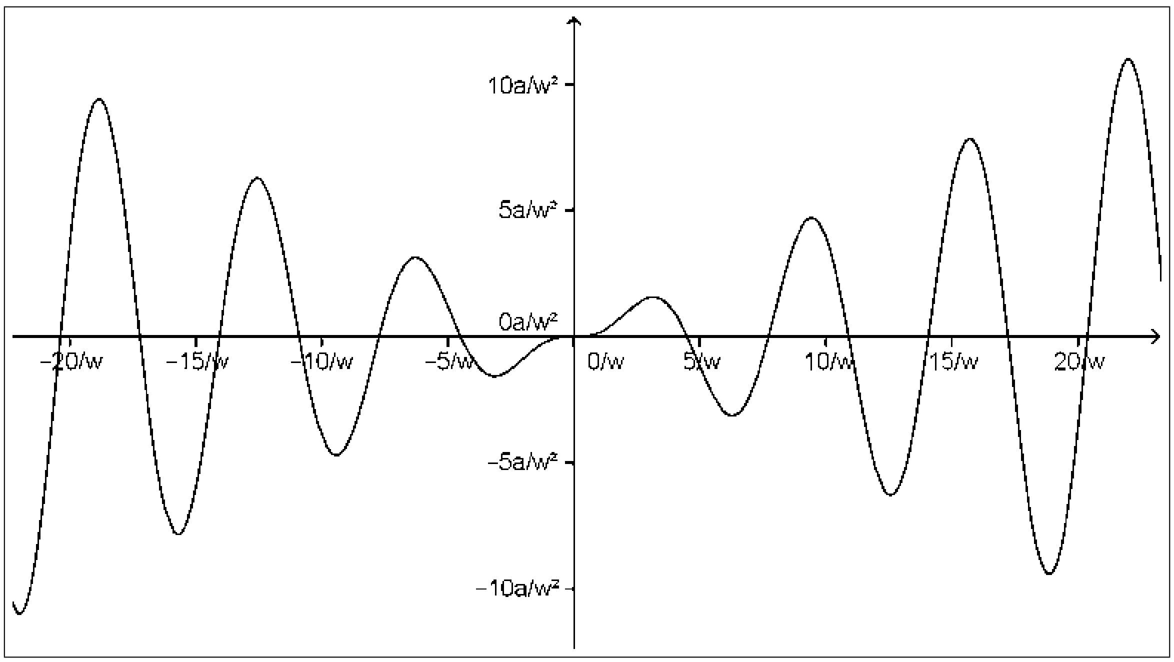

Figure 7.1: Plot of $y = \frac{a}{2w^2}\sin(wt) - \frac{a}{2w}t\cos(wt)$.